Very stable G-Higgs bundles

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Summary

This project is concerned with the study of **very stable G-Higgs bundles**, a special class of G-Higgs bundles parameterising closed, $\mathbb{C}^\times$-invariant Lagrangian affine subvarieties of the moduli space $\mathcal{M}(G)$. These are objects of interest from the point of view of the mirror symmetry of $\mathcal{M}(G)$ and provide a relevant collection of Lagrangians for which the study of their dual branes can be undertaken. Our current goal is to classify and describe these Higgs bundles.

Higgs bundles

Let G be a connected semisimple complex Lie group (or $\mathrm{GL}_n(\mathbb{C})$) with Lie algebra $\mathfrak{g}$. A **G-Higgs bundle** over a smooth projective complex curve C with canonical line bundle K_C **is a pair** (E, φ) where

- E is a principal G-bundle over C,
- φ is a section of $E(\mathfrak{g}) \otimes K_C$.

Example. If $G = \mathrm{GL}_n(\mathbb{C})$, we may regard E as a rank n vector bundle and $\varphi : E \to E \otimes K_C$ as a bundle morphism. As a recurring example, we consider

$$E_\delta := \mathcal{O}_C \oplus K_C^{-1}(D_1) \oplus K_C^{-2}(D_1 + D_2) \oplus \cdots \oplus K_C^{-n+1}(D_1 + \cdots + D_{n-1})$$

$$\varphi_\delta := \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \delta_1 & 0 & \dots & 0 & 0 \\ 0 & \delta_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{n-1} & 0 \end{pmatrix}$$

where $\delta := \{\delta_1, \dots, \delta_{n-1}\}$ are sections of $\mathcal{O}_C(D_i)$ whose divisor D_i is effective.

There is a **moduli space** of isomorphism classes of polystable G-Higgs bundles $\mathcal{M}(G)$. It is a **quasiprojective complex variety** and posseses a natural **holomorphic symplectic structure** ω on its smooth locus which is in fact **Hyperkähler**. With respect to this structure, it also features a **completely integrable system** $h_G : \mathcal{M}(G) \to \mathcal{A}(G)$ (here $\mathcal{A}(G)$ is a vector space) known as the **Hitchin system**.

Mirror symmetry

Mirror symmetry emerged from theoretical physics in the early 90s as a duality of quantum field theories. Mathematically, this translates into a statement of **duality** between **Calabi–Yau manifolds** X and $\hat{X}$ **interchanging** aspects of the **symplectic geometry** with those of the **complex geometry**.
In particular, moduli spaces of Higgs bundles $\mathcal{M}(G)$ exhibit this behaviour due to their Hyperkähler nature. Hausel and Thaddeus proposed that the corresponding dual is $\mathcal{M}(G^\vee)$, where $G^\vee$ is the **Langlands dual group** of G, and exhibited in some cases a duality of fibrations

$$\begin{tikzcd} \mathcal{M}(G) \arrow[dr, "h_G"] & & \mathcal{M}(G^\vee) \arrow[dl, "h_{G^\vee}"'] \\ & \mathcal{A}(G) \simeq \mathcal{A}(G^\vee) & \end{tikzcd}$$

agreeing with the proposal of Strominger, Yau and Zaslow that generically **the fibres** over a given point **are dual** special Lagrangian tori.
Kapustin–Witten propose an **enhanced duality**: first, there should be an equivalence between the derived category of **coherent sheaves** $\mathcal{D}^b(\mathcal{M}(G^\vee))$, encoding the complex geometry, and the **Fukaya category** $\mathrm{Fuk}(\mathcal{M}(G))$, encoding the symplectic geometry. This is an instance of Kontsevich's **Homological mirror symmetry**.
Then, considering the Hyperkähler structure, there should be a correspondence

$$\left\{ \begin{array}{c} \text{Complex Lagrangian} \\ \text{subvarieties of } \mathcal{M}(G) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Hyperholomorphic} \\ \text{sheaves on } \mathcal{M}(G^\vee) \end{array} \right\}.$$

Actions of $\mathbb{C}^\times$ and very stable G-Higgs bundles

Let X be a variety with an action of $\mathbb{C}^\times$. In other words, there is a notion of scaling points $x \in X$ by nonzero complex numbers $\lambda \in \mathbb{C}^\times$. We will assume that there is always a limiting point $l_x = \lim_{\lambda \to 0} \lambda x$. Such a limit point is fixed: $l_x = \lambda l_x$.
Given a fixed point $y \in X^{\mathbb{C}^\times}$ we consider its **upward flow**:

$$W_y^+ := \{x \in X : l_x = y\} \subseteq X.$$

These provide a decomposition of X studied by Białynicki-Birula.
Interestingly, the moduli spaces $\mathcal{M}(G)$ of G-Higgs bundles have such an action given by $(E, \varphi) \mapsto (E, \lambda\varphi)$ and the corresponding upward flows **are complex Lagrangian subvarieties**, providing examples where mirror symmetry can be studied.
Motivated by this, Hausel and Hitchin gave the following definition.

Definition. A smooth fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ is **very stable** if $W^+_{(E,\varphi)}$ is closed.

Goal. Can we classify/describe very stable G-Higgs bundles?

Regular Higgs field

Consider a smooth fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ with **generically regular** φ, i.e. such that the dimension of its centraliser is generically the minimum possible.

Example. For $G = \mathrm{GL}_n(\mathbb{C})$ such a fixed point is of the form $(E_\delta, \varphi_\delta)$ defined in the example to the left. Hausel and Hitchin proved that it is **very stable** if and only if the **divisor** $D_1 + \cdots + D_{n-1}$ is **reduced**.

In general, such a fixed point possesses a reduction of structure group to a maximal torus $T \subseteq G$ such that φ is a section of $E(\bigoplus_{j=1}^s \mathfrak{g}_{\alpha_j}) \otimes K_C$, where $\{\alpha_1, \dots, \alpha_r\}$ is a choice of simple roots. This **extra structure** allows to **define sections** δ_j of $E(\mathfrak{g}_{\alpha_j}) \otimes K_C$ as in the previous example. Denote by $\omega_j^\vee$ the corresponding fundamental coweights (dual to the simple roots).

Theorem (arXiv:2503.01289)**.** *The fixed point (E, φ) is **very stable** if and only if the **coweight***

$$\mu_c := \sum_{j=1}^r (\mathrm{ord}_c\, \delta_j)\omega_j^\vee$$

***is minuscule** for every $c \in C$. That is, if and only if $\alpha(\mu_c) \in \{0, \pm 1\}$ for every root α.*

This result was partly conjectured by Hausel and Hitchin after the study of an invariant called **virtual equivariant multiplicity** that we also compute. The main ingredient in the proof is the technique of Hecke transformations of G-Higgs bundles.

Nonregular Higgs field

There is **work in progress** towards the classification in the nonregular cases. For example, if $G = \mathrm{GL}_n(\mathbb{C})$ Peón-Nieto proves that fixed points of the form $E = V \oplus W$ with $\varphi : V \to W \otimes K_C$ are **not very stable** if $\operatorname{rank} V + \operatorname{rank} W > 3$.

We can use the recently developed **Toledo invariant**, which is a **topological invariant** $\tau(E, \varphi)$ associated to **every fixed point** $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$. This allows to get a **partial result** in the **most nonregular case** (besides $\varphi = 0$), i.e. when E only reduces to a Levi subgroup of a maximal parabolic subgroup of G (as in the case worked by Peón-Nieto above). If (E, φ) is such a point, we have the following.

Theorem. *There is an **explicit value** $\tau' \in \mathbb{R}$ such that*

$$\tau(E, \varphi) < \tau'$$

*implies that (E, φ) is **not very stable**.*

This value is attained by many components, ruling out very stable Higgs bundles in them. There is work in progress to expand this result to more points.

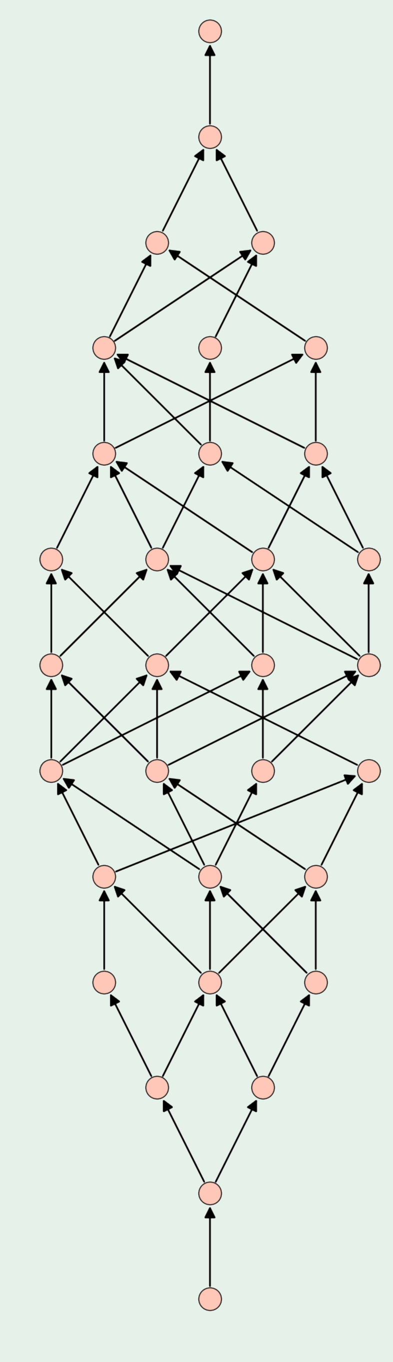
Parabolic Higgs bundles

Mirror symmetry is also present in moduli spaces of **strongly parabolic Higgs bundles**. Roughly speaking, these are defined by the following additional considerations:

- A choice of **punctures** $\{c_1, \dots, c_s\} \subseteq C$ and certain **parabolic weights** $\alpha_1, \dots, \alpha_s \in \mathfrak{t}$ for a maximal torus $T \subseteq G$.
- A choice of **parabolic structures**, which can be identified with elements $Q_1, \dots, Q_s \in G/P_{\alpha_j}$ of flag varieties for G.
- A **compatibility condition** on φ with the Q_i, plus the possibility of having **poles** at the c_i.

It is then reasonable to define very stable parabolic G-Higgs bundles in the same way. We have the following.

Theorem. *Let (E, φ, Q) be fixed under the $\mathbb{C}^\times$-action in the moduli space of strongly parabolic G-Higgs bundles with **generically regular** φ and **generic weights** α_j. There is a natural identification $Q_j = w_j \in G/B$ of the parabolic structures with **Weyl group elements** in the **flag variety**. They lie in a **Springer fibre** defined by $\varphi|_{c_j}$. If they are **not maximal under the Bruhat order** within that Springer fibre, then (E, φ, Q) is **not very stable**.*



A Springer fibre in type $C4$ with its Bruhat order depicted.