Wobbly parabolic G-Higgs bundles and affine flag varieties

Miguel González (ICMAT)

Geometric Structures and Stability
ICTS Bengaluru

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INSTITUTO DE CIENCIAS MATEMÁTICAS

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Motivation

- Mirror symmetry phenomena in moduli of **strongly parabolic G-Higgs bundles** $\mathcal{M}_{sp}(G,\alpha)$.
- **First**: **usual** moduli of G-**Higgs bundles** $\mathcal{M}(G)$.
- $\mathcal{M}(G)$ is Hyperkähler, holomorphic symplectic $\omega \in H^0(\Omega^2_{\mathcal{M}(G)})$.
- **Hitchin system**, duality of abelian fibrations: **SYZ mirror symmetry**.

$$\begin{tikzcd} \mathcal{M}(G) \arrow[dr, "h_G"] & & \mathcal{M}(G^\vee) \arrow[dl, "h_{G^\vee}"'] \\ & \mathcal{A}(G) \simeq \mathcal{A}(G^\vee) & \end{tikzcd}$$

(Hausel–Thaddeus, Donagi–Pantev)

Motivation II

- Duality $\mathrm{Fuk}(\mathcal{M}(G)) \simeq D^b_c(\mathcal{M}(G^\vee))$: **Homological mirror symmetry/geometric Langlands correspondence**. Classical limit:
$$D^b_c(\mathcal{M}(G)) \simeq D^b_c(\mathcal{M}(G^\vee))$$
Donagi–Pantev: generically Fourier–Mukai transform along fibres.
- Kapustin–Witten: hyperkähler **enhancements**. *BAA*-branes (local systems on complex Lagrangians) $\leftrightarrow$ *BBB*-branes (hyperholomorphic connections).
- **Upward flows**: $\mathbb{C}^\times$-invariant BAA-branes. Simplest: **very stable**. (Hausel–Hitchin, 2022; Peón–Nieto, 2024; G., 2025).
- What about **parabolic G-Higgs bundles**?

Mirror symmetry for strongly parabolic Higgs bundles

- **Moduli** space $\mathcal{M}(G, \alpha)$ is **Poisson**, **symplectic leaf** $\mathcal{M}_{sp}(G, \alpha)$ defined by **strongly parabolic**.
- **Mirror symmetry** in $\mathcal{M}_{sp}(G, \alpha)$, dual $\mathcal{M}_{sp}(G^\vee, \alpha^\vee)$. Physics in (Gukov–Witten, 2006).
- **Topological** mirror symmetry for $G = \mathrm{SL}_2(\mathbb{C}), \mathrm{SL}_3(\mathbb{C})$ (Gothen–Oliveira, 2019), later generalised (Su–Wang–Wen, 2022; Shen, 2023) to $\mathrm{SL}_n(\mathbb{C})$ using the p-adic integration techniques in (Gröchenig–Wyss–Ziegler, 2020).
- **SYZ** mirror symmetry (Biswas–Dey, 2012 for $G = \mathrm{SL}_n(\mathbb{C})$):

$$\begin{tikzcd}
\mathcal{M}_{sp}(G, \alpha) \arrow[dr, "h_G"'] & & \mathcal{M}_{sp}(G^\vee, \alpha) \arrow[dl, "h_{G^\vee}"] \\
& \mathcal{A}(G) \simeq \mathcal{A}(G^\vee) &
\end{tikzcd}$$

- **Goal:** Study **BAA-branes (upward flows)** in $\mathcal{M}_{sp}(G, \alpha)$.

Parabolic G-Higgs bundles

- G connected semisimple complex alg. group, Lie algebra $\mathfrak{g}$.
- Fix **maximal torus** and **Borel** $T \subseteq B \subseteq G$, $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$.
- **Simple roots** $\{\beta_1, \dots, \beta_r\} \subseteq \mathfrak{t}^*$.
- C smooth projective complex **curve**, canonical K_C.
- Fix **punctures** $D = c_1 + \cdots + c_s$ in C.
- Fix parabolic **weights** $\alpha_i \in \mathfrak{t}_{\mathbb{R}}$ at each c_i.
- Weights in **standard Weyl alcove** i.e. $\beta_j(\alpha_i) > 0$, $\delta(\alpha_i) < 1$ for δ the highest root.

Definition

A **parabolic G-Higgs bundle** (E, φ, Q) over C:

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C(D)$ (**Higgs field**)
- Parabolic structures $Q_i \in E|_{c_i}/B$. Determine $\mathfrak{q}_i \subseteq E(\mathfrak{g})|_{c_i}$.
- Compatibility: $\varphi|_{c_i} \in \mathfrak{q}_i \otimes K_C(D)$.

If every $\varphi|_{c_i}$ is nilpotent: **strongly parabolic**.

Example $G = \mathrm{PGL}_n(\mathbb{C})$

If $G = \mathrm{PGL}_n(\mathbb{C})$:

- E is a rank n vector bundle over C (up to line bundle twisting)
- α_i is given by real numbers

$$\alpha_i^1 > \cdots > \alpha_i^n$$

  with $\sum_j \alpha_i^j = 0$ and $\alpha_i^1 - \alpha_i^n < 1$.
- Each Q_i is a full flag

$$E_i^0 = 0 \subsetneq E_i^1 \subsetneq \cdots \subsetneq E_i^n = E|_{c_i}.$$

- $\varphi : E \to E \otimes K_C(D)$ is a traceless morphism such that

$$\varphi(E_i^j) \subseteq E_i^{j-1} \otimes K_C(D)$$

  (strongly parabolic).

$\mathbb{C}^\times$-action

- Natural $\mathbb{C}^\times$-action:

$$\lambda \cdot (E, \varphi, Q) = (E, \lambda\varphi, Q)$$

- **Semiprojective:** proper fixed point locus, limits at 0 exist.

Definition

The **upward flow** of the fixed point $\mathcal{E} := (E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$:

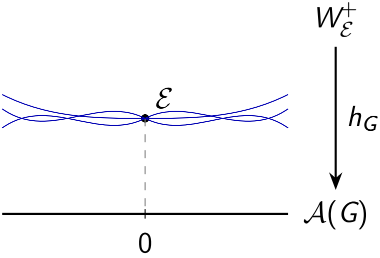
$$W^+_{\mathcal{E}} := \left\{ (E', \varphi', Q') : \lim_{\lambda \to 0} (E', \lambda\varphi', Q') = \mathcal{E} \right\} \subseteq \mathcal{M}_{sp}(G, \alpha)$$

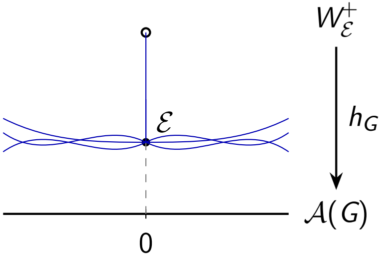
- $\mathcal{E}$ smooth $\rightsquigarrow$ $W^+_{\mathcal{E}}$ complex Lagrangian (BAA-brane).

Very stable parabolic G-Higgs bundles

Two possibilities for $\mathcal{E} := (E, \varphi, Q) \in \mathcal{M}_{sp}(G,\alpha)^{s\mathbb{C}^\times}$ (Hausel–Hitchin, 2022):

$W^+_{\mathcal{E}} \cap h_G^{-1}(0) = \{\mathcal{E}\}$	$W^+_{\mathcal{E}}$ has other nilpotents
$W^+_{\mathcal{E}}$ closed	$W^+_{\mathcal{E}}$ not closed
$h_G\|_{W^+_{\mathcal{E}}}$ finite cover	$\dim W^+_{\mathcal{E}} \cap h_G^{-1}(0) \geqslant 1$
$\mathcal{E}$ **very stable**	$\mathcal{E}$ **wobbly**





Fixed points $G = \mathrm{PGL}_2(\mathbb{C})$

Example with $G = \mathrm{PGL}_2(\mathbb{C})$. Take $D = c$, $k \geqslant 0$. Consider

$$E_k := \mathcal{O}_C \oplus K_C^{-1}((k-1)c)$$

$$\varphi_k := \begin{pmatrix} 0 & 0 \\ s_c^k & 0 \end{pmatrix}.$$

Fixed point if flag Q is either:

$$Q_{\mathrm{tail}} := 0 \subsetneq K_C^{-1}((k-1)c)|_c \subsetneq E_k|_c$$

or:

$$Q_{\mathrm{head}} := 0 \subsetneq \mathcal{O}_C|_c \subsetneq E_k|_c$$

(if $k > 0$).

Flag curves in $G = \mathrm{PGL}_2(\mathbb{C})$

When $k = 0$, only compatible flag is Q_{tail} and $(E_0, \varphi_0, Q_{\mathrm{tail}})$ is **very stable** (Hitchin section).

When $k > 0$, can choose **any** line ($t \in \mathbb{P}^1_{\mathbb{C}}$)

$$Q_t := 0 \subsetneq t \cdot \mathcal{O}_C|_c + K_C^{-1}((k-1)c)|_c \subsetneq E_k|_c$$

to get a parabolic Higgs bundle (E_k, φ_k, Q_t).

Remark

$$(E_k, \lambda \cdot \varphi_k, Q_t) \simeq (E_k, \varphi_k, Q_{\lambda t}).$$

Therefore, $(E_k, \varphi_k, Q_0) = (E_k, \varphi_k, Q_{\mathrm{tail}})$ is **wobbly**.

In this family, $(E_1, \varphi_1, Q_{\mathrm{head}})$ is the other **very stable**.

Expectation in $G = \mathrm{PGL}_n(\mathbb{C})$

- Let $G = \mathrm{PGL}_n(\mathbb{C})$, $L := K(D)$. Fixed points with φ **generically regular:**

$$E = \mathcal{O} \oplus L^{-1}(D_1) \oplus L^{-2}(D_1 + D_2) \oplus \cdots \oplus L^{-n+1}(D_1 + \cdots + D_{n-1})$$

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varphi_1 & 0 & \dots & 0 & 0 \\ 0 & \varphi_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_{n-1} & 0 \end{pmatrix}.$$

- $\mathbb{C}^\times$-action on flag variety $E|_{c_i}/B \simeq G/B$, flags Q_i are **fixed**.

Expectation

(E, φ, Q) is very stable if and only if $D_1 + \cdots + D_{n-1}$ is reduced & *minimality condition* on each flag.

First condition is (Hausel–Hitchin, 2022). Related work of (Henke, PhD thesis, 2024).

Generically regular fixed points

- Focus on fixed points with φ **generically regular**.
- Set $\mathfrak{g}_1 = \bigoplus_{i=1}^{r} \mathfrak{g}_{\beta_i}$ sum of simple root spaces.
- The **Weyl group** of G is $W := N_G(T)/T$.
- Recall basis of **fundamental coweights** $\{\omega_1^\vee, \dots, \omega_r^\vee\} \subseteq \mathfrak{t}$ (dual to $\{\beta_1, \dots, \beta_r\} \subseteq \mathfrak{t}^*$).

Proposition

Fixed points $(E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$ with **generically regular** φ:

- E reduces to a T-bundle E_T.
- $\varphi \in H^0(E_T(\mathfrak{g}_1) \otimes K_C(D))$.
- $Q_i \in W \cdot E_T|_{c_i}/(T \cap B) \subseteq E|_{c_i}/B$.

Twisted multiplicity divisor

Have **projection**

$$H^0(E_T(\mathfrak{g}_1) \otimes K_C(D)) \to H^0(E_T(\mathfrak{g}_{\beta_i}) \otimes K_C(D))$$

to get φ_i.
At $c_i \in D$, parabolic structure is

$$u_{c_i}(E, Q) \in W = W \cdot E_T|_{c_i}/(T \cap B).$$

Set $u_c(E, Q) = 1$ for $c \notin D$.

Definition

The **twisted multiplicity divisor** of (E, φ, Q) is

$$w(E, \varphi, Q) := \sum_{c \in C} \left(\sum_{i=1}^{r} -\mathrm{ord}_c(\varphi_i)\omega_i^\vee \right) u_c(E, Q) \cdot c$$

Coefficients in $\widetilde{W} := P^\vee \rtimes W$ the **extended affine Weyl group**.
When $c \notin D$, coefficient contained in $P_-^\vee \simeq W \backslash \widetilde{W}/W$.

Classification theorem

There is a **partial ordering** in $\widetilde{W}$ (**Bruhat order**). Minimal elements: **length zero**, i.e. $\mu \cdot u \in P^\vee \rtimes W$ such that

- μ is **antidominant minuscule** (i.e. $\beta(\mu) \in \{0, -1\}$ for positive roots $\beta \in \Delta^+$)
- u is the **maximum** element for the **Bruhat order** of W within the minimum length representatives in $W_{J_\mu} \backslash W$, for $W_{J_\mu} \subseteq W$ generated by simple root reflections fixing μ.

All coefficients of $w(E, \varphi, Q)$ minimal (i.e. minuscule when $c \notin D$, length zero when $c \in D$) $=:$ **reduced**.

Theorem

Smooth fixed point (E, φ, Q) with generically regular φ is **very stable** if and only if $w(E, \varphi, Q)$ is **reduced**.

Remark: If $D = 0$, recovers (G., 2025; arXiv:2503.01289).

Proof sketch

Strategy

Work in the **affine flag variety**

$$\mathrm{Fl}_G := G((z))/\mathcal{I}$$

where $\mathcal{I} \subseteq G[[z]]$ is the **standard Iwahori subgroup** defined by projecting to $B \subseteq G$ when $z = 0$.

- At $c \in D$, fixed point (E, φ, Q) defines **affine Springer fibre** $\mathrm{Fl}_G^{\varphi,c} \subseteq \mathrm{Fl}_G$, has **induced $\mathbb{C}^\times$-action**.
- Via **Hecke transformations**, $W^+_{(E,\varphi,Q)} \cap h_G^{-1}(0)$ at $c \in C$ given by $W^+_{\mathrm{Id}_G} \subseteq \mathrm{Fl}_G^{\varphi,c}$.
- $w(E, \varphi, Q)|_c$ length zero: $W^+_{\mathrm{Id}_G} = \mathrm{Id}_G$.
- $w(E, \varphi, Q)|_c$ not length zero: $W^+_{\mathrm{Id}_G}$ contains **curve**.
- Need to consider **stability**.
- Similar analysis over $c \notin D$ with $\mathrm{Gr}_G := G((z))/G[[z]]$.

Strategy: Hecke transformations

Definition

A **Hecke transformation** of (E, φ, Q) at $c \in C$ is another (E', φ', Q') with isomorphism over $C_0 := C \setminus c$:

$$(E', \varphi', Q')|_{C_0} \xrightarrow{\sim} (E, \varphi, Q)|_{C_0}$$

How to obtain at $c \in D$:

1. Fix formal disk $C_1 \simeq \operatorname{Spec} \mathbb{C}[[z]]$ at c. Set $C_{01} := C_0 \cap C_1$.
2. Local data
   $(E|_{C_0}, Q|_{C_0}, C_1 \times G, B \in G/B, \Phi : C_{01} \times G \xrightarrow{\sim} E|_{C_{01}})$.
3. Modify $\Phi \mapsto \Phi \circ \sigma$ for $\sigma \in \operatorname{Aut}(C_{01} \times G) \simeq G((z))$.

If σ lifts to $\operatorname{Aut}(C_1 \times G, B) \simeq \mathcal{I}$, result is isomorphic.

$$\mathrm{Fl}_G := G((z))/\mathcal{I}.$$

Hecke transformations II

- Higgs field φ becomes $(\varphi_0 \in H^0(E|_{C_0}(\mathfrak{g}) \otimes K_C(D)), \varphi_1 \in \mathfrak{i})$ compatible with Φ and $Q|_{C_0}$.
- Hecke transformation: $(\varphi_0, \varphi_1) \mapsto (\varphi_0, \mathrm{Ad}_{\sigma^{-1}}(\varphi_1))$.
- Well defined? **Affine Springer fibre**

$$\mathrm{Fl}_G^{\varphi_1} := \{\sigma \in \mathrm{Fl}_G : \mathrm{Ad}_{\sigma^{-1}}(\varphi_1) \in \mathfrak{i}\} \subseteq \mathrm{Fl}_G .$$

Proposition

(E, φ, Q) fixed point with gen. reg. φ, there is trivialisation of $(E|_{C_1}, Q|_c)$ and $\mathbb{C}^\times$-action:

$$\lambda \cdot \sigma := \exp\left(\sum_{i=1}^{r} -\log(\lambda)\omega_i^\vee\right) \cdot \sigma$$

on $\mathrm{Fl}_G^{\varphi_1}$ such that Hecke transformation $(E'_\lambda, \varphi'_\lambda, Q'_\lambda)$ satisfies

$$(E'_\lambda, \varphi'_\lambda, Q'_\lambda) \simeq (E'_{\lambda=1}, \lambda\varphi'_{\lambda=1}, Q'_{\lambda=1}).$$

Thank you!