Departamento de Matemáticas, Facultad de Ciencias
Universidad Autónoma de Madrid

Cyclic Higgs bundles and Vinberg theory

Master's thesis

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Author: Miguel González González
Director: Oscar García Prada
Tutor: Mario García Fernández

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Abstract

We apply the theory of finite order automorphisms of a semisimple Lie group developed by Vinberg to the study of cyclic Higgs bundles, which correspond to the fixed points of actions of finite cyclic groups on the moduli space of Higgs bundles. For some of the resulting spaces, we introduce a topological invariant which extends the previously existing Toledo invariant for the case of spaces of fixed points by actions of $\mathbb{Z}/2\mathbb{Z}$ of Hermitian type as well as for the spaces of fixed points by the $\mathbb{C}^*$-action. We prove a bound for this invariant and we exhibit a rigidity phenomenon when the bound is attained. Finally, we explore certain aspects of the Hitchin fibration on the spaces of cyclic Higgs bundles, giving a description of the generic fibres in a selected family of examples.

Keywords. Cyclic Higgs bundle, Vinberg pair, moduli space, cyclic grading of a Lie algebra, Toledo invariant, Hitchin fibration.

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Contents

CHAPTER 1
Introduction

Higgs bundles were introduced by Hitchin in 1987 [34], appearing naturally as solutions of the self-duality equations on a Riemann surface, a dimensional reduction of the instanton physics Yang-Mills equation in four dimensions. For a complex semisimple Lie group G with Lie algebra $\mathfrak{g}$ and a compact Riemann surface X of genus $g \geq 2$, they can be defined as pairs (E, φ) consisting of a holomorphic principal G-bundle E over X and a holomorphic section φ of the vector bundle $E(\mathfrak{g}) \otimes K_X$, where $E(\mathfrak{g})$ is the vector bundle associated to E via the adjoint representation of G in $\mathfrak{g}$ and $K_X = T^*X$ is the holomorphic cotangent bundle of X.

Since their introduction, these objects have been extensively studied, showcasing many interesting properties. There are stability notions for a Higgs bundle, and the *moduli space of polystable Higgs bundles*, $\mathcal{M}(G)$, has been shown to possess a very rich topology and geometry. One of the major aspects is the existence of the *nonabelian Hodge correspondence* [15, 18, 34, 47, 49], via which $\mathcal{M}(G)$ is homeomorphic to the variety of G-characters of $\pi_1(X)$, whose elements are the (completely reducible) representations $\rho : \pi_1(X) \to G$. Moreover, the smooth locus of $\mathcal{M}(G)$ has a hyperkähler structure, consequence of its interpretation as solution of the self-duality equations.

Furthermore, there exists [32] a completely integrable system $\mathcal{M}(G) \to \mathcal{A}$, known as the *Hitchin system*, which maps the moduli space onto an affine space with the generic fibres being abelian varieties. This fibration can be used to relate the moduli space $\mathcal{M}(G)$ to *mirror symmetry* and *Langlands duality*, central aspects in current mathematics. It has been a key ingredient of the proof for the Fundamental Lemma of the Langlands program by Ngô, for which he was awarded a Fields medal. This system also has a distinguished section [33], commonly referred to as *Hitchin section* (which is unique if G is adjoint), whose image is a component in $\mathcal{M}(G^{\mathbb{R}})$, the moduli space of $G^{\mathbb{R}}$-Higgs bundles, where $G^{\mathbb{R}} \subseteq G$ is the split real form of G. This component is called *Hitchin component* and it generalises Teichmüller space.

Our goal is to study the theory of Higgs bundles that arises in relation to finite order automorphisms θ of the group G (see [20] for a recent exploration of this idea, as well as [25]). Such an automorphism induces a grading of the Lie algebra $\mathfrak{g}$ by a finite cyclic group $\mathbb{Z}/m\mathbb{Z}$, where m is the order of θ, and the fixed point subgroup $G^\theta \leq G$, which is reductive with Lie algebra $\mathfrak{g}_0$, acts via the adjoint representation on each piece $\mathfrak{g}_i$ of the grading. The resulting pairs $(G^\theta, \mathfrak{g}_i)$ were originally studied by Vinberg [50], and are called *Vinberg θ-pairs* in consequence. We will also consider other groups with the same Lie algebra $\mathfrak{g}_0$, such as the connected component of the identity $G_0 \subseteq G^\theta$, or G_θ, the G-normaliser of G^θ.

Given a Vinberg θ-pair $(G^\theta, \mathfrak{g}_i)$, it is possible to use the fact that $G^\theta \leq G$, $\mathfrak{g}_i \subseteq \mathfrak{g}$ and G^θ acts on $\mathfrak{g}_i$ to define, in an analogous manner to Higgs bundles, objects denominated $(G^\theta, \mathfrak{g}_i)$-*Higgs pairs*, which are pairs (E, φ) where E is a holomorphic principal G_0-bundle and φ is a holomorphic section of $E(\mathfrak{g}_i) \otimes K_X$. There also exist stability notions for these pairs, providing a well defined moduli space $\mathcal{M}(G^\theta, \mathfrak{g}_i)$. This space maps into $\mathcal{M}(G)$, and the G-Higgs bundles in the image are called *θ-cyclic Higgs bundles*. These will be our main object of study.

An important, extensively studied particular case of θ-cyclic Higgs bundles occurs when θ has order two. For the Vinberg θ-pairs $(G^\theta, \mathfrak{g}_1)$, there is a corresponding real form $G^\mathbb{R} \leq G$ such that the nonabelian Hodge correspondence can be extended to provide a homeomorphism between the moduli space $\mathcal{M}(G^\theta, \mathfrak{g}_1)$ and the variety of characters of $\pi_1(X)$ with values in the real Lie group $G^\mathbb{R}$. The resulting Higgs pairs are also called $G^\mathbb{R}$-Higgs bundles in consequence.

Another important reason for the study of θ-cyclic Higgs bundles is that they appear as fixed point subvarieties in $\mathcal{M}(G)$ of the action of finite cyclic groups [25]. This action is the one generated by $(E, \varphi) \mapsto (\theta(E), \zeta\theta(\varphi))$, for ζ a primitive m-th root of unity, where m is the order of θ. In the case of order two, the resulting subvarieties are lagrangian with respect to one of the Kähler structures on $\mathcal{M}(G)$.

One of the main results of Vinberg [50, 51] with regard to Vinberg θ-pairs $(G^\theta, \mathfrak{g}_i)$ is the fact that the invariant polynomial ring $\mathbb{C}[\mathfrak{g}_i]^{G^\theta}$ is isomorphic to a (finitely generated) polynomial ring. This makes it possible to define a Hitchin map $\mathcal{M}(G^\theta, \mathfrak{g}_i) \to \mathcal{A}$ onto an affine base in analogy to the aforementioned Hitchin system. This map has been studied in the order two case [23, 29, 42, 44, 45], but its study in the higher order case has not yet been approached.

Cyclic Higgs bundles have also appeared in different contexts in the literature. They were first introduced in [48], where Simpson refers to them as *cyclotomic harmonic bundles*, and they are used to construct local systems with specific properties. In [3], Baraglia defines cyclic Higgs bundles as certain subspace of the Hitchin component, which coincides with the θ-cyclic Higgs bundles as explained above inside of said component. The motivation for this is that they constitute solutions for the *affine Toda equations*, from which extra properties of these bundles can be deduced. The same Higgs bundles have also been studied in [17] from the point of view of harmonic maps and in [39] in the case of non-compact Riemann surfaces. In [37], cyclic Higgs bundles in the Hitchin component (as before) are related to *cyclic surfaces*, which can be used to prove the existence of minimal surfaces in certain rank 2 symmetric spaces. These ideas have also been exploited in more recent work [13] to parametrise certain class of holomorphic curves in the pseudosphere of dimension 6. As mentioned above, in [25] it is shown that θ-cyclic Higgs bundles appear as fixed points subvarieties in $\mathcal{M}(G)$ of the action of finite cyclic groups. In [14], θ-cyclic Higgs bundles for θ of inner type are considered, providing (among other results) a parametrisation of certain components of $\mathcal{M}(\mathrm{SO}_0(n, n+1))$ generalising the Hitchin component and undetected by the main topological invariants of the space.

We explore mainly two aspects of cyclic Higgs bundles. The first one is relevant in the case where the cyclic grading of the Lie algebra $\mathfrak{g}$ lifts to a $\mathbb{Z}$-grading satisfying certain properties. Such a lift always exists in the case, among many others, of inner automorphisms of $\mathrm{SL}_n(\mathbb{C})$. For this situation we define a topological invariant, the *Toledo invariant*, which generalises the Toledo invariant existing for the order two case [6] (the corresponding real forms $G^{\mathbb{R}}$ for which a good lifting $\mathbb{Z}$-grading exists are called of *hermitian type* as the associated symmetric space is hermitian) as well as for Higgs bundles associated to a $\mathbb{Z}$-grading of the Lie algebra [5].

For this Toledo invariant we prove a bound, known in the aforementioned particular cases as the *Arakelov–Milnor–Wood inequality*, which is the content of Theorem 8. In the case of real forms of hermitian type of a special kind, called *tube type*, there exist rigidity results [6] for the locus of Higgs bundles attaining the bound. We extend these results to the analogue of tube type in Vinberg θ-pairs, called *JM-regular*. This occurs in the form of the *Cayley correspondence* which establishes that the locus of cyclic Higgs bundles whose Toledo invariant attains the bound injects into the moduli space of (C, V)-Higgs bundles for a different pair (C, V) associated to a subgroup $C \leq G$ and a subspace $V \subseteq \mathfrak{g}$. We also show that the map is surjective in the case where (C, V) is a Vinberg θ'-pair for a different automorphism θ' on a different group G'. This is the content of Theorem 9.

The second main aspect that was explored is the Hitchin map. For the order two case, it is known [23] that the abelianness of the generic fibre occurs when the associated real form $G^{\mathbb{R}}$ satisfies the condition of being *quasi-split*. We start by proposing, in Definition 23, a natural extension of quasi-split applying to any Vinberg θ-pair. Then, for a particular class of quasi-split inner Vinberg pairs of $G = \mathrm{SL}_n(\mathbb{C})$, we extend the results of Schaposnik [45] to give a description of the generic fibres of the Hitchin map, using *spectral curves*, from which it can be checked that they are abelian, and also it can be used to provide a different proof for the bounds on the Toledo invariant in this case. This is collected in Proposition 12 and Proposition 14.

The document is structured as follows. Chapter 2 is devoted to the introduction of the Lie theoretical ingredients that will play a role in the definition and study of cyclic Higgs bundles and their moduli spaces. First, in Section 2.1, we collect the theory of $\mathbb{Z}$-gradings of Lie algebras following [36, Chapter X]. Even though at first cyclic Higgs bundles do not seem to be directly related to these gradings, their study will be necessary as a main tool for some of the results relating to them. Then, in Section 2.2 we introduce cyclic gradings of Lie algebras and some of the aspects of the theory of Vinberg θ-pairs. For both types of gradings, we explain their existing classification in terms of labellings of *Dynkin diagrams* and *Kac diagrams*. Three of the main references for the chapter are [51, Section 3.7], [27, Chapter 3] and [50].

In Chapter 3 we introduce the main definitions of the general theory of Higgs bundles over a compact Riemann surface and their moduli spaces. First, in Section 3.1, we define Higgs pairs for a representation of a Lie group, which we motivate by examining properties of the resulting objects for particular choices of groups and representations. We will also introduce the notions of stability that are required in order to define moduli spaces with desirable geometric properties. Next, in Section 3.2 we explain how the fixed points of a natural $\mathbb{C}^*$-action on the moduli space consists of Higgs pairs associated to a certain $\mathbb{Z}$-grading. Finally, in Section 3.3 we define cyclic Higgs bundles and explain their relation with Vinberg θ-pairs.

Then, in Chapter 4 we introduce a topological invariant for moduli spaces of $(G_0, \bar{\mathfrak{g}}_1)$-Higgs pairs, where $(G_0, \bar{\mathfrak{g}}_1)$ is a Vinberg θ-pair coming from a *special* $\mathbb{Z}$-grading as explained in Section 2.2. It generalizes the Toledo invariant for moduli spaces of $G^{\mathbb{R}}$-Higgs bundles where $G^{\mathbb{R}}$ is a real form of hermitian type of G introduced in [6], which was motivated by previous studies in particular cases such as $G^{\mathbb{R}} = \mathrm{SU}(p,q)$ in [10]. It also generalizes the Toledo invariant for moduli spaces of $(G_0, \mathfrak{g}_1)$-Higgs pairs associated to a prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ introduced in [5]. In section 4.1 we explain how the invariant is defined, in terms of the theory of prehomogeneous vector spaces, using the fact that the Vinberg θ-pair comes from a $\mathbb{Z}$-grading. We also compute the value of the invariant in the case of cyclic quivers presented in Example 5. In section 4.2 we establish the *Arakelov–Milnor–Wood inequality*, which shows that the Toledo invariant can only take values in an interval bounded below. Finally, in Section 4.3 we explore the resulting moduli space when the Toledo invariant attains the bound from the previous inequality. We use ideas of previous works [5, 6, 9, 10].

Chapter 5 is devoted to the study of the Hitchin map, a fibration of the moduli space $\mathcal{M}(G_0, \mathfrak{g}_i)$ for Vinberg θ-pairs $(G_0, \mathfrak{g}_i)$ over a vector space $\mathcal{A}$. In Section 5.1 we will define the Hitchin map and explore some of its features such as the existence of a section in the case of G-Higgs bundles. In Section 5.2 we will see why in the case of G-Higgs bundles the generic fibre is an abelian variety, by describing what this fibre is using the *spectral correspondence*. For the case of Higgs bundles of real forms, it is known [23] that this good behaviour on the fibers arises when the form is *quasi-split*, so in Section 5.3 we give a candidate definition for quasi-split Vinberg θ-pairs and compute some examples of such pairs. In Section 5.4 we show how a spectral correspondence for the real form $\mathrm{SU}(k,k)$ established by Schaposnik [45] can be generalized to the case of cyclic quiver bundles where each piece has the same rank, and we derive in this situation the Arakelov–Milnor–Wood inequality proven in previous chapter as a consequence of this description.

Finally, in Chapter 6 we collect the conclusions of this document and discuss possible future lines of work.

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CHAPTER 2
Lie theory background

2.1. $\mathbb{Z}$-gradings and Vinberg $\mathbb{C}^*$-pairs

Throughout this section we consider a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$.

Definition 1. A **$\mathbb{Z}$-grading** of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ is a decomposition as a direct sum of vector subspaces

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j,$$

such that $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$.

Notice that, since $\mathfrak{g}$ is finite-dimensional, $\mathfrak{g}_j = 0$ for all but finitely many $j \in \mathbb{Z}$. Also, $\mathfrak{g}_0$ is a Lie subalgebra, as $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$. Moreover, every element in $\mathfrak{g}_j$ for $j \neq 0$ is *nilpotent*, in other words, given $e \in \mathfrak{g}_j$, the map $\mathrm{ad}(e) : \mathfrak{g} \to \mathfrak{g}$ given by $X \mapsto [e, X]$ is nilpotent. Indeed, if $X \in \mathfrak{g}_k$, we have that $\mathrm{ad}(e)^r(X) \subseteq \mathfrak{g}_{rj+k}$ so for large enough r it must become zero.

The existence of a $\mathbb{Z}$-grading is equivalent to a $\mathbb{C}^*$-action on $\mathfrak{g}$ by Lie algebra automorphisms.

Proposition 1. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra. There is a bijection between $\mathbb{Z}$-gradings of $\mathfrak{g}$ and group homomorphisms $\gamma : \mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g})$.*

Proof. Given a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ and $\lambda \in \mathbb{C}^*$, define $\gamma_\lambda : \mathfrak{g} \to \mathfrak{g}$ by $\gamma_\lambda|_{\mathfrak{g}_j} = \lambda^j \,\mathrm{Id}_{\mathfrak{g}_j}$. It is a Lie algebra automorphism because if $X \in \mathfrak{g}_j$, $Y \in \mathfrak{g}_k$, we have $\gamma_\lambda[X, Y] = \lambda^{j+k}[X, Y] = [\lambda^j X, \lambda^k Y] = [\gamma_\lambda X, \gamma_\lambda Y]$. The map $\gamma : \lambda \mapsto \gamma_\lambda$ is clearly a group homomorphism.

On the other hand, given a group homomorphism $\gamma : \mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g})$, we have the decomposition $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ in weight spaces $\mathfrak{g}_j = \{X \in \mathfrak{g} : \gamma(\lambda)X = \lambda^j X\}$, which give a $\mathbb{Z}$-grading due to the fact that each $\gamma(\lambda)$ is compatible with the Lie bracket. □

We have the following result [36, Lemma 10.15].

Proposition 2. *Let* $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ *be a* $\mathbb{Z}$*-graded complex semisimple Lie algebra. Then, there exists a grading element, that is,* $\zeta \in \mathfrak{g}_0$ *such that* $\mathfrak{g}_j = \{X \in \mathfrak{g} : [\zeta, X] = jX\}$.

Proof. Consider the linear map $D : \mathfrak{g} \to \mathfrak{g}$ given by $D|_{\mathfrak{g}_k} = k\,\mathrm{Id}_{\mathfrak{g}_k}$. Take $X \in \mathfrak{g}_j$ and $Y \in \mathfrak{g}_k$. Then $D[X, Y] = (k + j)[X, Y] = [DX, Y] + [X, DY]$. Thus D is a derivation of $\mathfrak{g}$. The only derivations of a semisimple Lie algebra are the ones of the form $\mathrm{ad}(Z)$ for $Z \in \mathfrak{g}$ [36, Proposition 1.121], thus $D \equiv \mathrm{ad}(\zeta)$. As $D(\zeta) = [\zeta, \zeta] = 0$, we have $\zeta \in \mathfrak{g}_0$. □

Notice that we always have $[\mathfrak{g}_0, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$. This phenomenon has the following consequence: suppose that G is some semisimple complex Lie group with algebra $\mathfrak{g}$, and $G_0 \leq G$ is the connected subgroup corresponding to $\mathfrak{g}_0$. Then G_0 acts on $\mathfrak{g}_j$ by restriction of the adjoint action on $\mathfrak{g}$. This action, even though it is not necessarily transitive (that is, $\mathfrak{g}$ is not a *homogeneous vector space*), has an open orbit.

Definition 2. Let G be a complex reductive Lie group and $\rho : G \to \mathrm{GL}(V)$ a (holomorphic) representation. If there exists an open G-orbit $\Omega \subseteq V$, the vector space V is called a **prehomogeneous vector space** for G.

In this case it can be seen [36, Proposition 10.1] that the open orbit is dense and unique. Typically we will also say that the pair (G, V) is a prehomogeneous vector space.

Theorem 1 (Vinberg [36, Theorem 10.19])**.** *Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ a $\mathbb{Z}$-grading. Let $G_0 \leq G$ be the centralizer of the grading element ζ. Then G_0 is reductive and $(G_0, \mathfrak{g}_1)$ is a prehomogeneous vector space.*

Remark 1. The theorem above works with the first graded piece, $\mathfrak{g}_1$. However, given $j \neq 0$, the Lie algebra $\mathfrak{g}$ has a $\mathbb{Z}$-graded subalgebra given by $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{kj}$ whose first piece is $\mathfrak{g}_j$ and whose zeroth piece is $\mathfrak{g}_0$, to which we can apply the previous theorem to conclude that $(G_0, \mathfrak{g}_j)$ is a prehomogeneous vector space. This is why, in general, we will mostly be concerned with prehomogeneous vector spaces (arising from $\mathbb{Z}$-gradings) of the form $(G_0, \mathfrak{g}_1)$.

These prehomogeneous vector spaces coming from $\mathbb{Z}$-gradings will be of special interest, so we give the following definition.

Definition 3. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ a $\mathbb{Z}$-grading. Let $G_0 \leq G$ be the centralizer of the grading element ζ. The prehomogeneous vector spaces $(G_0, \mathfrak{g}_k)$ (for $k \neq 0$) are called **Vinberg $\mathbb{C}^*$-pairs**.

Definition 4. Let (H, W) and (G, V) be two prehomogeneous vector spaces. If $H \subseteq G$ is a subgroup, $W \subseteq V$ is a vector subspace, and the action of H in W is obtained by restricting the action of G on V, we say that (H, W) is a **prehomogeneous vector subspace** of (G, V).

The last necessary concept relates to the following standard theorem about nilpotent elements on Lie algebras (that is, elements $e \in \mathfrak{g}$ such that $\mathrm{ad}(e)$ is nilpotent):

Theorem 2 (Jacobson–Morozov, [36, Theorem 10.3])**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $e \in \mathfrak{g}$ a nonzero nilpotent element. Then, there exist $h, f \in \mathfrak{g}$ such that $[h, e] = 2e$, $[e, f] = h$ and $[h, f] = -2f$, that is, the triple (h, e, f), called $\mathfrak{sl}_2$**-triple**, spans a Lie subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{C}$.*

Moreover, for any $h \in \mathfrak{g}$ with $[h, e] = 2e$ and $h \in \mathrm{ad}(e)(\mathfrak{g})$, it is possible to find a unique $f \in \mathfrak{g}$ with the previous properties.

This implies that, denoting by e', h', f' the generators of $\mathfrak{sl}_2\mathbb{C}$ satisfying $[h', e'] = 2e'$, $[e', f'] = h'$, $[h', f'] = -2f'$, there is a Lie algebra representation $\mathfrak{sl}_2\mathbb{C} \to \mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$ given by $e' \mapsto \mathrm{ad}(e)$, $h' \mapsto \mathrm{ad}(h)$ and $f' \mapsto \mathrm{ad}(f)$.

Going back to $\mathbb{Z}$-gradings and the associated prehomogeneous vector space $(G_0, \mathfrak{g}_1)$, as $\mathfrak{g}_1$ consists only of nilpotent elements, we can first ask in which graded pieces do the associated h and f belong. We have the following result [36, Lemma 10.18].

Proposition 3. *Let $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ be a graded complex semisimple Lie algebra and suppose that $e \in \mathfrak{g}_1$ is nonzero. Then it is possible to choose the $\mathfrak{sl}_2$-triple (h, e, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$.*

Proof. We already know that e is nilpotent, so by Theorem 2 we obtain an $\mathfrak{sl}_2$-triple (h, e, f). Write $h = \sum_{k\in\mathbb{Z}} h_k$ and $f = \sum_{k\in\mathbb{Z}} f_k$ the decompositions according to the grading. Then, $2e = [h, e] = \sum_{k\in\mathbb{Z}}[h_k, e]$ implies that $[h_0, e] = 2e$ (and $[h_j, e] = 0$ for $j \neq 0$). Similarly, from $h = [e, f] = \sum_{k\in\mathbb{Z}}[e, f_k]$ we get that $[e, f_{-1}] = h_0$. In other words, we can use the last part of Theorem 2 with h_0, so that there exists an $\mathfrak{sl}_2$-triple (h_0, e, f').

Finally, writing $f' = \sum_{k\in\mathbb{Z}} f'_k$ the decomposition according to the grading, we have $[e, f'_{-1}] = h_0$ as before, but also $-2f' = [h_0, f'] = \sum_{k\in\mathbb{Z}}[h_0, f'_k]$, so that $[h_0, f'_{-1}] = -2f'_{-1}$. Hence (h_0, e, f'_{-1}) is the desired triple. □

Thus, it is possible to take an element $e \in \mathfrak{g}_1$ and complete it to (h, e, f) with $h \in \mathfrak{g}_0$. Then, $\mathrm{ad}(\frac{h}{2})$ also fixes e, so we can compare the original grading to the one one having grading element $\mathrm{ad}(\frac{h}{2})$. It will be technically convenient when the two agree, motivating the following definition from [5].

Definition 5. Let $(G_0, \mathfrak{g}_1)$ be the prehomogeneous vector space with open orbit $\Omega \subseteq \mathfrak{g}_1$ associated to a $\mathbb{Z}$-grading of a complex semisimple Lie algebra $\mathfrak{g}$ with grading element ζ. We say that it is **Jacobson–Morozov regular** or **JM-regular** if there exists an $\mathfrak{sl}_2$-triple (h, e, f) with $e \in \Omega$ such that $\zeta = \frac{h}{2}$.

In this case, it is a consequence of Malcev–Kostant theorem [36, Theorem 10.10] that every element $e \in \Omega$ can be included into such a triple $(2\zeta, e, f)$. Also, in the above situation, if $e \in \Omega$ is chosen and (h, e, f) is any $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$ (not necessarily the one with $h = 2\zeta$), then h is conjugate to 2ζ under the adjoint action of G_0, by [5, Proposition 2.19].

We will now see some examples of prehomogeneous vector spaces.

Example 1. The vector space $\mathbb{C}^n$ is a prehomogenous vector space for the standard representation of $\mathrm{GL}_n(\mathbb{C})$. Indeed, for any nonzero $v, w \in \mathbb{C}^n$ we can find many invertible transformations $A \in \mathrm{GL}_n(\mathbb{C})$ with $Av = w$. On the other hand $A0 = 0$ for any $A \in \mathrm{GL}_n(\mathbb{C})$. Thus the action has two orbits, $\Omega = \mathbb{C}^n \setminus \{0\}$ and $\{0\}$, the first of which is open.

Example 2. We will analyse in detail one of the main examples that we use later. We take $G = \mathrm{SL}_n(\mathbb{C})$, the determinant one invertible linear transformations of a complex n-dimensional vector space V. We fix a direct sum decomposition $V = V_0 \oplus \cdots \oplus V_{m-1}$, where if $\dim V_i = d_i$ then $\sum_i d_i = n$. Then, the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ consists of traceless endomorphisms of V, which we grade by

$$\mathfrak{g}_k := \left(\bigoplus_{j=0}^{m-1} \mathrm{Hom}(V_j, V_{j+k}) \right)_0 ,$$

where $V_j = 0$ for $j \notin \{0, \ldots, m-1\}$ and the subscript zero means taking the subspace of traceless endomorphisms, and is only meaningful for $\mathfrak{g}_0$. This is alternatively defined by the grading element $\zeta \in \mathfrak{g}_0 = (\bigoplus_{j=0}^{m-1} \mathrm{End}(V_j))_0$ given by $\zeta|_{V_j} = (j - \alpha)\,\mathrm{Id}|_{V_j}$, where α is a fixed constant (depending on the d_i) so that the obtained map is traceless.

The associated prehomogeneous vector space is $(G_0, \mathfrak{g}_1)$, where

$$G_0 = S(\mathrm{GL}_{d_0}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{d_{m-1}}(\mathbb{C})),$$

$$\mathfrak{g}_1 = \bigoplus_{j=0}^{m-1} \mathrm{Hom}(V_j, V_{j+1}).$$

The space $\mathfrak{g}_1$ consists of representations of a quiver with m vertices and arrows $i \mapsto i + 1$ for $i \in \{0, \ldots, m-2\}$ (a *linear quiver*, or *type A_m quiver*, the latter as a reference to Dynkin diagram types) where we put V_i on the i-th vertex:

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{m-2}} V_{m-1} .$$

The orbits of the action of G_0 on such representations have been studied and are explained, for example, in [1, Section 2] (more precisely, they study the action of $\prod \mathrm{GL}_{d_i}(\mathbb{C})$, but it can be checked that it has the same orbits as that of G_0). For an element $f \in \mathfrak{g}_1$ as in the previous diagram, and $0 \leq i < j \leq m-1$, we denote $r_{ij} := \mathrm{rank}(f_{j-1} \circ \cdots \circ f_i)$ the rank of the consecutive composition, a linear map from V_i to V_j. Then, each feasible choice of ranks $(r_{ij})_{0 \leq i < j \leq m-1}$, with $0 \leq r_{ij} \leq \min\{d_i, \ldots, d_j\}$, determines a unique orbit. The open orbit is the one where ranks are maximal, that is, $r_{ij} = \min\{d_i, \ldots, d_j\}$ for all i, j.

Finally, let us explain how to determine whether this prehomogeneous vector space is JM-regular. An element of the open orbit $\Omega \in \mathfrak{g}_1$ is given by $e = (e_0, \ldots, e_{m-1})$ where each $e_j : V_j \to V_{j+1}$ is of maximal rank. For example, we can choose a basis $B_j = \{v_1^j, \ldots, v_{d_j}^j\}$ for each V_j and let e_j be the one with matrix $\begin{pmatrix} Id_{d_{j+1}} & 0 \end{pmatrix}$ or $\begin{pmatrix} Id_{d_j} \\ 0 \end{pmatrix}$, depending on whether $d_j \leq d_{j+1}$. In order to complete e to an $\mathfrak{sl}_2$-triple, we can do so from the Jordan blocks of e. This is because if $\{u_0, \dots, u_{s-1}\}$ is a basis for a Jordan block (that is, $e(u_k) = u_{k+1}$ for $k < s-1$, $e(u_s) = 0$, and $\mathrm{Im}(e) \cap \langle u_0 \rangle = 0$), then one can define a traceless linear map h by $h(u_j) = -(s-1-2j)u_j$. Doing this on each Jordan block gives $h \in \mathfrak{g}$ with $[h, e] = 2e$ and the remaining f is defined as $f(u_j) = j(s-j)u_{j-1}$. Moreover, notice that by definition of e, we can partition the basis $\bigcup_{j=0}^{m-1} B_j$ into Jordan blocks. This means that the resulting h given by the previous method is actually in $\mathfrak{g}_0$.

As ζ is fixed by the action of G_0, the JM-regular cases are precisely whenever the h constructed above equals 2ζ. In short, given the dimensions d_j, the above procedure determines whether the space is JM-regular or not. An example of JM-regular case that can be verified with the previous method is when $d_j = d_{m-j-1}$ for $j \in \{0, \dots, c\}$ and $d_0 \leq d_1 \leq \cdots \leq d_c$, where $c = \lfloor \frac{m-1}{2} \rfloor$.

2.1.1. Classification of $\mathbb{Z}$-gradings

We will finish the section by exhibiting how every possible $\mathbb{Z}$-grading on $\mathfrak{g}$ can be classified, following [27]. Consider a $\mathbb{Z}$-grading $\gamma : \mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g})$. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra. As the grading element ζ is semisimple (i.e. $\mathrm{ad}(\zeta)$ diagonalises), we can assume that $\zeta \in \mathfrak{t}$ (see, for example, [36, Proposition 2.13]). Select a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$ with $\alpha_i(\zeta) \geq 0$ (otherwise replace α_i by $-\alpha_i$). As $\mathrm{ad}(\zeta)$ has integral eigenvalues, we also have that $p_i := \alpha_i(\zeta) \in \mathbb{Z}$. Let $\Delta \subseteq \mathfrak{t}^*$ be the root system for $\mathfrak{t} \subseteq \mathfrak{g}$ and recall the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [T, X] = \alpha(T) \cdot X, T \in \mathfrak{t}\}.$$

Since $\mathrm{ad}(\zeta)$ acts with eigenvalue p_i on $\mathfrak{g}_{\alpha_i}$, we have with respect to the $\mathbb{Z}$-grading that $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{g}_{p_i}$. Moreover, the bracket relation for root spaces

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{if } \alpha + \beta \notin \Delta, \alpha + \beta \neq 0 \\ \text{a one-dimensional subspace of } \mathfrak{t} & \text{if } \alpha + \beta = 0 \end{cases}$$

gives that for any root $\alpha = \sum_{i=1}^{r} k_i \alpha_i$ we have $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_p$ with $p = \sum_{i=1}^{r} k_i p_i$. As the Cartan subalgebra is abelian and $\zeta \in \mathfrak{t}$, we also have $\mathfrak{t} \subseteq \mathfrak{g}_0$.

If we now choose a different $\mathbb{Z}$-grading, given by the element ζ', it could be that $\zeta' \notin \mathfrak{t}$ or that $\alpha_i(\zeta') < 0$ for some i, but there always exists [27, Section 3.3.5] an automorphism $a \in \mathrm{Aut}(\mathfrak{g})$ with $a(\zeta') \in \mathfrak{t}$ and with $\alpha_i(a(\zeta')) \geq 0$. This automorphism is of the form $\exp(\mathrm{ad}(X))$ for some $X \in \mathfrak{g}$, i.e. an **inner automorphism**. The group of inner automorphisms, $\mathrm{Int}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$, is precisely the identity component of $\mathrm{Aut}(\mathfrak{g})$. The quotient $\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g})$ is called the group of **outer automorphisms**. As $a(\zeta')$ satisfies the desired conditions, we can apply the same analysis as above with respect to the roots of $\mathfrak{t}$ to the grading given by $a(\zeta')$.

Thus, every $\mathbb{Z}$-grading, up to inner automorphism, is given by numbering each of the simple roots Π with a non-negative integer $p_i \in \mathbb{Z}_{\geq 0}$. Conversely, any such numbering results in a $\mathbb{Z}$-grading where we set $\mathfrak{g}_p = \bigoplus_{\alpha\in\Delta_p} \mathfrak{g}_\alpha$ for $p \neq 0$, where $\Delta_p = \{\alpha = \sum k_i\alpha_i : \sum k_i p_i = p\}$ and $\mathfrak{g}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta_0} \mathfrak{g}_\alpha$ with Δ_0 defined analogously.

Recall finally that, since $\mathfrak{g}$ is semisimple, it has an associated **Dynkin diagram** which is a tree whose vertices are the elements of Π and an edge between α_i and α_j exists if and only if $\alpha_i + \alpha_j \in \Delta$ (the edge has multiplicity depending on the angle between α_i and α_j in $\mathfrak{t}^*$ with respect to the dual of the Killing form). Thus, $\mathbb{Z}$-gradings are classified (up to inner automorphism) by labellings of the Dynkin diagram vertices with non-negative integers.

Example 3. We will exhibit the numbering of the Dynkin diagram for the $\mathbb{Z}$-grading in Example 2. In this example, we have $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ so that the Dynkin diagram is of type A_{n-1}:

$$\bullet \text{ --- } \bullet \text{ --- } \dots \text{ --- } \bullet \text{ --- } \bullet,$$

which is a linear tree with $n-1$ vertices. As $\zeta \in \mathfrak{g}_0$ is given by $\zeta|_{V_j} = (j-\alpha)\,\mathrm{Id}_{V_j}$, it is a diagonal matrix (we work with respect to a basis of V obtained by concatenating bases of $V_0,\dots,V_{m-1}$ in that order) so that we can take $\mathfrak{t} \subseteq \mathfrak{g}$ to be the subalgebra of diagonal (and traceless) $n \times n$ matrices. Letting $e_i \in \mathfrak{t}^*$ be the linear form that reads the i-th entry of the diagonal, we choose as simple roots $\alpha_i = e_{i+1} - e_i$ for $i \in \{1, \dots, n-1\}$. These work because the entries on the diagonal of ζ are non decreasing, so $\alpha_i(\zeta) \geq 0$.

Recall that the root spaces with respect to these simple roots α_i are $\mathfrak{g}_{\alpha_i} = \mathbb{C} \cdot E_i$, where E_i is the matrix whose only non-zero entry is a 1 at the $(i+1)$-th row and i-th column, hence sending the i-th basis vector to the $(i+1)$-th basis vector. Thus, it is an endomorphism of some V_j (so that $E_i \in \mathfrak{g}_0$) unless $i \in \{d_0, d_0 + d_1, d_0 + d_1 + d_2, \dots, \sum_{l=0}^{m-2} d_l\}$, since if $i = \sum_{l=0}^{k} d_l$ then $E_i \in \mathrm{Hom}(V_k, V_{k+1}) \in \mathfrak{g}_1$. This shows that the corresponding labelling of the Dynkin diagram has a zero on each vertex except for the ones at positions of the form $\sum_{l=0}^{k} d_l$ for $k \in \{0, \dots, m-2\}$ where it has a one. The following is an example for the $d_0 = 1, d_1 = 3, d_2 = 2$ case, where the simple roots are ordered left to right:

$$\bullet^1 \text{ --- } \bullet^0 \text{ --- } \bullet^0 \text{ --- } \bullet^1 \text{ --- } \bullet^0.$$

2.2. $\mathbb{Z}/m\mathbb{Z}$-gradings and Vinberg θ-pairs

In the same way as above, we can define cyclic gradings on semisimple Lie algebras.

Definition 6. Let $m \in \mathbb{N}$ and let $\mathfrak{g}$ be a semisimple complex Lie algebra. A $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$ is a decomposition as a direct sum of vector subspaces

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j,$$

such that $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$.

Although the definition is similar, there are fundamental differences with respect to $\mathbb{Z}$-gradings. For example, it is no longer true that every element in $\mathfrak{g}_j$ for $j \neq 0$ is nilpotent. As in the previous section, we can view gradings in terms of automorphisms:

Proposition 4. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra. There is a bijection between $\mathbb{Z}/m\mathbb{Z}$-gradings of $\mathfrak{g}$ and group homomorphisms $\gamma : \mu_m \to \mathrm{Aut}(\mathfrak{g})$, where $\mu_m = \{z \in \mathbb{C}^* : z^m = 1\}$ is the subgroup of m-th roots of unity. These homomorphisms are also in bijection with order m automorphisms $\theta \in \mathrm{Aut}_m(\mathfrak{g})$.*

Proof. The last assertion is a consequence of μ_m being the cyclic group of order m. Fix $\zeta \in \mathbb{C}^*$ a primitive m-th root of unity. From $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j$, we obtain an order m automorphism θ by the rule $\theta|_{\mathfrak{g}_j} \equiv \zeta^j \,\mathrm{Id}_{\mathfrak{g}_j}$. This is well defined and of order m, both because $\zeta^m = 1$. It is compatible with the bracket: if $X \in \mathfrak{g}_j$ and $Y \in \mathfrak{g}_k$, we have $\theta[X, Y] = \zeta^{j+k}[X, Y] = [\zeta^j X, \zeta^k Y] = [\theta X, \theta Y]$.

Conversely, given $\theta \in \mathrm{Aut}_m(\mathfrak{g})$, we obtain a $\mathbb{Z}/m\mathbb{Z}$-grading by taking the eigenspace decomposition, that is, setting $\mathfrak{g}_j = \{X \in \mathfrak{g} : \theta X = \zeta^j X\}$. We can do so because the minimal polynomial is $\theta^m - 1$ which has distinct roots. The fact that it is a $\mathbb{Z}/m\mathbb{Z}$-grading comes from the compatibility of θ with the Lie bracket. □

In all the situations we consider there is a complex semisimple Lie group G with Lie algebra $\mathfrak{g}$, and the $\theta \in \mathrm{Aut}_m(\mathfrak{g})$ lifts to an order m automorphism of G, that is, there is some $\tilde{\theta} \in \mathrm{Aut}_m(G)$ whose tangent map at 1_G is θ. In what follows we denote by θ both automorphisms, as the context will make clear which of the two we are referring to. This assumption is automatic for simply connected G, and in any event it will be satisfied in our cases of interest.

Now let $G_0 \leq G$ be the connected subgroup corresponding to the Lie algebra $\mathfrak{g}_0$. In fact, for what follows it is also possible to take in place of G_0 any closed subgroup $G' \leq G_\theta$ (where G_θ is the G-normaliser of the fixed point group G^θ) satisfying $G_0 \leq G'$. For example, the choice of G^θ is more natural from the point of view of cyclic Higgs bundles, as we will see in Section 3.3. In any case, the chosen group is reductive with Lie algebra $\mathfrak{g}_0$ and, as in the $\mathbb{Z}$-grading case, the adjoint representation restricted to each graded piece gives an action $G' \to \mathrm{Aut}(\mathfrak{g}_j)$, from the fact that $[\mathfrak{g}_0, \mathfrak{g}_j] = \mathfrak{g}_j$.

Definition 7. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\theta \in \mathrm{Aut}_m(G)$. The pair $(G_0, \mathfrak{g}_1)$ constructed as above is called a **Vinberg θ-pair**.

Remark 2. As in Remark 1, it is enough to consider the action of G_0 on $\mathfrak{g}_1$. This is because the subalgebra $\bigoplus_{j\in\mathbb{Z}/m'\mathbb{Z}} \mathfrak{g}_{jk}$ is a $\mathbb{Z}/m'\mathbb{Z}$-graded algebra with zero-th piece $\mathfrak{g}_0$ and first piece $\mathfrak{g}_k$, for $m' = \frac{m}{(m,k)}$.

A key aspect for considering Vinberg θ-pairs is the good structure of the G_0-invariant polynomial functions on $\mathfrak{g}_1$, which will permit some of the geometric constructions in Chapter 5. We denote by $\mathbb{C}[\mathfrak{g}_1]^{G_0} = \{p \in \mathrm{Sym}(\mathfrak{g}_1^*) : p(\mathrm{Ad}(g)(x)) = p(x), x \in \mathfrak{g}_1, g \in G_0\}$ the ring of invariant polynomial functions. In order to say more about its structure, we need the following definitions:

Definition 8. Let $(G_0, \mathfrak{g}_1)$ be a Vinberg θ-pair. A **Cartan subspace** is a vector subspace $\mathfrak{c} \subseteq \mathfrak{g}_1$, maximal with the following properties: $[\mathfrak{c}, \mathfrak{c}] = 0$, and each $X \in \mathfrak{c}$ verifies that $\operatorname{ad} X$ is diagonalizable.

Definition 9. Let $(G_0, \mathfrak{g}_1)$ be a Vinberg θ-pair with Cartan subspace $\mathfrak{c}$. Let $N_{G_0}(\mathfrak{c}) = \{g \in G_0 : \operatorname{Ad}(g)(\mathfrak{c}) \subset \mathfrak{c}\}$ be the normaliser of $\mathfrak{c}$ under the G_0-action, and $C_{G_0}(\mathfrak{c}) = \{g \in G_0 : \operatorname{Ad}(g)(X) = X, X \in \mathfrak{c}\}$ be the centraliser. The **little Weyl group** is defined by $W(\mathfrak{c}) = N_{G_0}(\mathfrak{c})/C_{G_0}(\mathfrak{c})$.

Then, we have the following main result of [50].

Theorem 3. *Let $(G_0, \mathfrak{g}_1)$ be a Vinberg θ-pair with Cartan subspace $\mathfrak{c}$. Then, restriction of polynomial functions from $\mathfrak{g}_1$ to $\mathfrak{c}$ defines a ring isomorphism $\mathbb{C}[\mathfrak{g}_1]^{G_0} \to \mathbb{C}[\mathfrak{c}]^{W(\mathfrak{c})}$.*

Moreover, $W(\mathfrak{c})$ is a finite group acting by complex reflections by an hyperplane, and $\mathbb{C}[\mathfrak{c}]^{W(\mathfrak{c})} \simeq \mathbb{C}[\mathfrak{g}_1]^{G_0}$ is a polynomial ring, that is, it is generated freely by some finite subset of homogeneous polynomials, $\{f_1, \dots, f_r\} \subset \mathbb{C}[\mathfrak{g}_1]^{G_0}$, where $r = \dim \mathfrak{c}$ is called the rank *of the pair.*

Remark 3 (Saturation)**.** The previous theorem is proven assuming that G_0 is the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. A $\mathbb{Z}/m\mathbb{Z}$-grading $\theta \in \operatorname{Aut}_m(G)$ is said to be **saturated** if the invariant ring $\mathbb{C}[\mathfrak{g}_1]^{G'}$ is the same for any choice of closed subgroup $G_0 \leq G' \leq G_\theta$. If $\mathfrak{c} \subseteq \mathfrak{g}_1$ is a Cartan subspace, Vinberg [50] characterises saturated automorphisms as those where every class in G_θ/G_0 contains an element in the centraliser $C_{G_\theta}(\mathfrak{c})$.

There are sufficient conditions for an automorphism $\theta \in \operatorname{Aut}_m(G)$ to be saturated, given in [50]. For example, if m is prime, or $\varphi(m) \dim \mathfrak{c} = \operatorname{rank} \mathfrak{g}$ (here φ is the Euler function), or G is classical ($\mathrm{SL}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C}), \mathrm{Sp}_n(\mathbb{C})$) except in the case where $G = \mathrm{SL}_{2k}(\mathbb{C})$, θ is non-inner (as defined in Section 2.2.1) and $4|m$, then θ is saturated.

Even if the pair is not saturated, some groups between G_0 and G_θ can still result in the same invariants. For example, if G is simply connected then $G_0 = G^\theta$.

A very relevant first example for Vinberg θ-pairs is that related to real forms of G.

Example 4. A *real form* of a complex reductive Lie group G is the fixed point locus of an antiholomorphic involution $\sigma : G \to G$. For example, if $G = \mathrm{GL}_n(\mathbb{C})$ and $\sigma : A \mapsto \bar{A}$, we get $G^\sigma = \mathrm{GL}_n(\mathbb{R})$ as a real form. Let $G^{\mathbb{R}}$ be a real form of G and $\mathfrak{g}$ be the Lie algebra of G. The theory of Cartan [36, Chapter VI] gives an involution $\theta \in \operatorname{Aut}_2(G)$ which induces a decomposition as a direct sum of vector spaces, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, such that $\mathfrak{h}$ is the Lie algebra of the complexification H of a maximal compact subgroup $H^{\mathbb{R}} \subseteq G^{\mathbb{R}}$. This yields the Vinberg θ-pair $(H, \mathfrak{m})$ from the real form $G^{\mathbb{R}}$. Note that this is an example for $m = 2$. These pairs are also called **symmetric pairs**.

The following will be one our main examples.

Example 5. As in the linear quiver representation case of Example 2, we take $G = \mathrm{SL}_n(\mathbb{C})$, seen as transformations of some n-dimensional vector space V. We once again fix a direct sum decomposition in m pieces, $V = V_0 \oplus \cdots \oplus V_{m-1}$, where $\dim V_i = d_i$. Then, we have an order m automorphism $\theta \in \mathrm{Aut}_m(G)$ defined by $\theta(g) = I_{d_0,\ldots,d_{m-1}} \cdot g \cdot I^{-1}_{d_0,\ldots,d_{m-1}}$, where the operator $I_{d_0,\ldots,d_{m-1}} \in \mathrm{GL}_n(\mathbb{C})$ is defined by $I_{d_0,\ldots,d_{m-1}}|_{V_j} \equiv \zeta^j \,\mathrm{Id}_{V_j}$ for $\zeta \in \mathbb{C}^*$ a primitive m-th root of unity. Then, the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ gets a $\mathbb{Z}/m\mathbb{Z}$ grading which is

$$\mathfrak{g}_k := \left(\bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} \mathrm{Hom}(V_j, V_{j+k}) \right)_0 ,$$

where the subscript zero means taking the subspace of traceless endomorphisms, and again is only meaningful for $\mathfrak{g}_0$.

The associated Vinberg θ-pair is $(G_0, \mathfrak{g}_1)$, where

$$G_0 = S(\mathrm{GL}_{d_0}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{d_{m-1}}(\mathbb{C})),$$

$$\mathfrak{g}_1 = \bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} \mathrm{Hom}(V_j, V_{j+1}).$$

The space $\mathfrak{g}_1$ consists of representations of a quiver with m vertices and arrows $i \mapsto i+1$ for $i \in \mathbb{Z}/m\mathbb{Z}$ (a *cyclic quiver*) where we put V_i on the i-th vertex:

$$\begin{tikzcd} V_0 \arrow[r, "f_0"] & V_1 \arrow[r, "f_1"] & \cdots \arrow[r, "f_{m-2}"] & V_{m-1} \arrow[lll, bend left, "f_{m-1}"] \end{tikzcd} .$$

We will conclude the example by computing a Cartan subspace $\mathfrak{c} \subseteq \mathfrak{g}_1$. Assume without loss of generality that d_0 is minimal (otherwise, rotate the pieces V_j). Fix a splitting $V_j = U_j \oplus W_j$ of each vector space such that $\dim U_j = d_0$, and a basis $B_j = \{v^j_1, \ldots, v^j_{d_0}\}$ for each U_j. Consider, for $\lambda = (\lambda_1, \ldots, \lambda_{d_0}) \in \mathbb{C}^{d_0}$, the element $f^\lambda = (f^\lambda_0, \ldots, f^\lambda_{m-1}) \in \mathfrak{g}_1$ defined by $f^\lambda_j(v^j_k) := \lambda_k v^{j+1}_k$, $f^\lambda_j|_{W_j} \equiv 0$. The collection $\mathfrak{c}' := \{f^\lambda : \lambda \in \mathbb{C}^m\}$ is a d_0-dimensional vector subspace of $\mathfrak{g}_1$ isomorphic to $\mathbb{C}^{d_0}$.

Moreover, every element $f^\lambda \in \mathfrak{c}'$ verifies that $\mathrm{ad}\, f^\lambda$ is diagonalizable. In order to see this, as we are working with a Lie algebra of endomorphisms, it suffices to see that f^λ is diagonalizable (as a consequence of the fact that a Jordan decomposition for f^λ gives a Jordan decomposition for $\mathrm{ad}\, f^\lambda$, see [36, Section 1.7]). But $v_{k,l} := v^0_k + \zeta^l v^1_k + \cdots + \zeta^{l(m-1)} v^{m-1}_k$ is an eigenvector with eigenvalue $\zeta^{-l}\lambda_k$. These give md_0 linearly independent eigenvectors spanning $\bigoplus_j U_j$, and outside of that $f^\lambda = 0$, so f^λ diagonalizes.

Finally, $[\mathfrak{c}', \mathfrak{c}'] = 0$, because $f^\lambda f^\beta - f^\beta f^\lambda = 0$, as it is clearly 0 on the W_j, and given $v^j_k \in U_j$ one has $(f^\lambda f^\beta - f^\beta f^\lambda)(v^j_k) = (\lambda_k\beta_k - \beta_k\lambda_k)v^{j+2}_k = 0$.

Thus, $\mathfrak{c}' \subset \mathfrak{c}$, where $\mathfrak{c}$ is a Cartan subspace. Furthermore, it is known [38, Theorem 1] that the invariant polynomial ring $\mathbb{C}[\mathfrak{c}]^{W(\mathfrak{c})}$ is generated freely by the coefficients of the characteristic polynomial of $f_{m-1} \circ \cdots \circ f_0 \in \mathrm{End}(V_0)$, thus $\dim \mathfrak{c} = \dim V_0 = d_0$, hence $\mathfrak{c}' = \mathfrak{c}$.

2.2.1. Classification of $\mathbb{Z}/m\mathbb{Z}$-gradings

As in the previous section, we will exhibit a classification of the $\mathbb{Z}/m\mathbb{Z}$ gradings of $\mathfrak{g}$ via numberings of some associated diagram, following [27]. We fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, its associated root system Δ and the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta$. By Proposition 4, a $\mathbb{Z}/m\mathbb{Z}$-grading is the same as an order m automorphism $\theta \in \mathrm{Aut}_m(\mathfrak{g})$, which is semisimple (i.e. diagonalisable) as it acts on $\mathfrak{g}_j$ via multiplication by ζ^j. We will first explain the classification of **inner** semisimple automorphisms of order m, that is $\theta \in \mathrm{Int}_m(\mathfrak{g})$ (recall from the end of Section 2.1 that inner automorphisms are those of the form $\exp(\mathrm{ad}(X))$ for some $X \in \mathfrak{g}$). Note that not every $\mathbb{Z}/m\mathbb{Z}$-grading comes from inner automorphisms.

First, we explain a classification theorem for semisimple elements in $\mathrm{Int}(\mathfrak{g})$. As $\mathfrak{g}$ is semisimple, we can decompose it as a sum of simple Lie algebras. If $\mathfrak{g}'$ is one of those summands, then $\theta(\mathfrak{g}')$ is another summand (it is a simple subalgebra, and the only possibility is that it is one of the summands by [36, Theorem 1.54]). Thus the Lie algebra $\mathfrak{g}$ splits as a direct sum of θ-invariant subalgebras of the form $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ where each $\mathfrak{s}_i$ is simple and $\theta(\mathfrak{s}_i) = \mathfrak{s}_{i+1}$ (with indices mod k), so we can restrict our attention to this case. Moreover, $\theta^k \in \mathrm{Aut}(\mathfrak{s}_1)$, θ^k is semisimple if and only if θ is, and the pair $(\mathfrak{s}, \theta)$ is equivalent to the data $(\mathfrak{s}_1, \theta^k, k)$. In other words, we have reduced the problem to that of a simple Lie algebra.

We can then assume that $\mathfrak{g}$ is simple and θ is a semisimple inner automorphism. As $G := \mathrm{Int}(\mathfrak{g})$ is the group of automorphisms of the form $\exp(\mathrm{ad}\, X)$ for $X \in \mathfrak{g}$, we have that the Lie algebra of G is identified with $\mathfrak{g}$. Let $\delta \in \Delta$ be the highest root, that is, the one verifying that if $\delta = \sum_{i=1}^r n_i\alpha_i$, the quantity $\sum_{i=1}^r n_i$ is maximal with respect to other roots. Set $\alpha_0 := -\delta$ the *lowest root*. We define the **extended system of simple roots** or **affine system of simple roots** $\Pi' := \{\alpha_0, \alpha_1, \dots, \alpha_r\}$. Note that, if we additionally set $n_0 := 1$, we have $\sum_{i=0}^r n_i\alpha_i = 0$. Loosely speaking, these play the role of a *barycentric reference system* for the roots of $\mathfrak{g}$.

Automorphisms of the simple roots Π (that is, isometries of Π with respect to the dual of the Killing form, equivalently, automorphisms of the Dynkin diagram of $\mathfrak{g}$) can be extended to automorphisms of the affine simple roots Π', giving the embedding $\mathrm{Aut}(\Pi) \leq \mathrm{Aut}(\Pi')$. One has [27, Section 3.3.6] that $\mathrm{Aut}(\Pi') = \Gamma \rtimes \mathrm{Aut}(\Pi)$, where $\Gamma \trianglelefteq \mathrm{Aut}(\Pi')$ is a normal subgroup isomorphic to $\pi_1(G)$ (recall that here $G = \mathrm{Int}(\mathfrak{g})$, if G' is the simply connected group with Lie algebra $\mathfrak{g}$, then $G = \mathrm{Ad}(G')$). The group Γ acts simply transitively on set of α_i that satisfy $n_i = 1$.

Any element $x \in \mathfrak{t}$ is determined by the barycentric coordinates $x_0 := 1 - \delta(x)$, $x_i := \alpha_i(x)$ for $i \in \{1, \dots, r\}$, which satisfy $\sum_{i=0}^r n_i x_i = 1$. We then have the classification theorem [27, Theorem 3.3.11]:

Theorem 4. *Let $\mathfrak{g}$ be a simple complex Lie algebra. Any inner semisimple automorphism of $\mathfrak{g}$ is, up to conjugation by an element in $\mathrm{Int}(\mathfrak{g})$, of the form $\exp(2\pi i x) \in \mathrm{Int}(\mathfrak{g})$, where $x \in \mathfrak{t}$ has barycentric coordinates $x_0, \dots, x_r$ satisfying both $Re(x_i) \geq 0$ and $Re(x_i) = 0 \implies \mathrm{Im}(x_i) \geq 0$. The automorphisms for x and x' are conjugate by an element in $\mathrm{Int}(\mathfrak{g})$ if and only if the barycentric coordinates of x can be taken to those of x' via the action of Γ, and they are conjugate by an element in $\mathrm{Aut}(\mathfrak{g})$ if and* *only if the barycentric coordinates of x can be taken to those of x' via the action of* $\mathrm{Aut}(\Pi')$.

Now, we shall use Theorem 4 to classify inner $\mathbb{Z}/m\mathbb{Z}$-gradings (that is, those given by an inner $\theta \in \mathrm{Int}_m(\mathfrak{g})$) up to inner automorphism. As before, we work with $\mathfrak{g}$ simple. We can then assume that $\theta = \exp(2\pi i x)$ for $x \in \mathfrak{t}$ with barycentric coordinates $x_0, x_1, \ldots, x_r$ satisfying the conditions of Theorem 4. We have that $\theta^m = \mathrm{Id}$, so that $x_i = \frac{p_i}{m}$ for non-negative integers $p_i \in \mathbb{Z}_{\geq 0}$. Since $\sum_{i=0}^r n_i x_i = 1$, we get the relation

$$\sum_{i=0}^{r} n_i p_i = m.$$

Thus, an inner $\mathbb{Z}/m\mathbb{Z}$-grading defines a labelling of the affine simple roots Π' with non-negative integer labels. Conversely, from such a labelling we can reconstruct the $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$. First, from the labels p_i we can retrieve $m = \sum_{i=0}^r n_i p_i$. Then, any root $\alpha \in \Delta$ can be represented uniquely as

$$\alpha = \sum_{i=0}^{r} k_i \alpha_i, \qquad 0 \leq k_i \leq n_i,$$

from which we obtain that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_p$ for $p = \sum_{i=0}^r k_i p_i$. The subalgebra $\mathfrak{g}_0$, as in the case of $\mathbb{Z}$-gradings, also contains the Cartan subalgebra $\mathfrak{t}$.

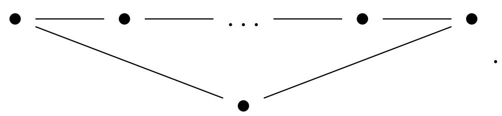
Remark 4. If the labels p_i have greatest common divisor d, the order of θ is $m' = \frac{m}{d}$ so that the $\mathbb{Z}/m\mathbb{Z}$-grading is actually coming from a $\mathbb{Z}/m'\mathbb{Z}$-grading with labels $p_i' = \frac{p_i}{d}$.

Then, as in the case of $\mathbb{Z}$-gradings, we have seen that every inner $\mathbb{Z}/m\mathbb{Z}$-grading is, up to inner automorphisms, given by a labelling of an associated diagram called **affine Dynkin diagram** of $\mathfrak{g}$. The labelled diagram is also sometimes referred to as **Kac diagram**. It is the diagram obtained by setting one vertex per α_i, connecting α_i and α_j if and only if $\alpha_i + \alpha_j \in \Delta$, and letting the multiplicity of the edge be determined by the angle between them with respect to the dual of the Killing form (in affine Dynkin diagrams, as opposed to usual Dynkin diagrams, the multiplicity can go up to 4).

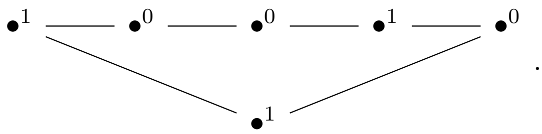
Example 6. We will identify the labelling associated to the cyclic grading in Example 5. First, let us construct the affine Dynkin diagram of the corresponding Lie algebra which is $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$. Start by recalling its (standard) Dynkin diagram:

$$\bullet \text{ —— } \bullet \text{ —— } \ldots \text{ —— } \bullet \text{ —— } \bullet.$$

There are $n-1$ vertices corresponding, left to right, to the simple roots in the system $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$. The highest root is $\delta = \alpha_1 + \cdots + \alpha_{n-1}$, so the lowest root is $\alpha_0 = -\alpha_1 - \cdots - \alpha_{n-1}$. This also means that $n_i = 1$ for all $i \in \{0, \ldots, n-1\}$. In the affine Dynkin diagram, the new vertex corresponding to α_0 will be connected to those α_i such that $\alpha_0 + \alpha_i$ is a root, namely α_1 and α_{n-1}. Hence, this is the affine Dynkin diagram of type A_{n-1}:



Its automorphism group is $\mathrm{Aut}(\Pi') = D_n$, the dihedral group of $2n$ elements. As explained in the general theory, we have $\mathrm{Aut}(\Pi') = \Gamma \rtimes \mathrm{Aut}(\Pi)$ for the normal subgroup $\Gamma \simeq \pi_1(\mathrm{Int}(\mathfrak{g})) = \pi_1(\mathrm{Ad}(\mathrm{SL}_n(\mathbb{C}))) = \pi_1(\mathrm{PSL}_n(\mathbb{C})) = \mathbb{Z}/n\mathbb{Z}$, which is the cyclic subgroup that *rotates* the vertices (also, for $n \geq 3$, $\mathrm{Aut}(\Pi) = \mathbb{Z}/2\mathbb{Z}$ acts by *reflecting* the vertices). With the same choices made in Example 3, a similar argument to the one in that example shows that the simple roots α_i for $i \geq 1$ are labelled with a zero, except for the ones at positions of the form $\sum_{l=0}^{k} d_l$ for $k \in \{0, \dots, m-2\}$, where it is labelled with a one. The lowest root space is $\mathfrak{g}_{\alpha_0} = \mathbb{C} \cdot E_0$, where E_0 is the matrix whose only nonzero entry is at the top right corner (row 1 and column n). Thus, $E_0 \in \mathrm{Hom}(V_{m-1}, V_0) \in \mathfrak{g}_1$, and it is labelled with a one. For example, the $d_0 = 1, d_1 = 3, d_2 = 2$ case results in the following Kac diagram:



We check that, as there is a vertex labelled with a one per each choice of $k \in \{0, \dots, m-2\}$ as well as in α_0, there are m ones, and for all $i \in \{0, \dots, n-1\}$ we have $n_i = 1$, so that indeed $\sum_{i=0}^{n-1} n_i p_i = m$.

Not every automorphism of finite order of $\mathfrak{g}$ is inner. It is known that $\mathrm{Out}(\mathfrak{g}) \simeq \mathrm{Aut}(\Pi)$, that is, the outer automorphism group is the same as the automorphisms of the Dynkin diagram, and thus $\mathrm{Out}(\mathfrak{g})$ can have either one, two or three elements (the latter only in D_4 type, corresponding to $\mathfrak{so}(8, \mathbb{C})$). We have seen that if θ is trivial in $\mathrm{Out}(\mathfrak{g})$, it corresponds to a numbering of the affine Dynkin diagram. If it is the other element (or the third one in D_4), there is an analogous (albeit harder to establish) correspondence [27, Section 3.3.11], [31, Section X.5] in terms of other kinds of *Kac diagrams*. To give an overview, the correspondence works exactly in the same way but replacing the affine Dynkin diagram of the system of simple roots of $\mathfrak{g}$ with the extended Dynkin diagram of a different system, called the *system of simple θ-roots* of $\mathfrak{g}$, where $\theta \in \mathrm{Out}(\mathfrak{g})$ is the desired outer class.

The θ-roots are constructed as follows. Pick $s \in \theta \cdot \mathrm{Aut}(\mathfrak{g})$ a semisimple automorphism. Let $T \leq \mathrm{Aut}(\mathfrak{g})^s$ be a maximal torus. Then, $S := \langle T, s \rangle \leq \mathrm{Aut}(\mathfrak{g})$ is a *quasitorus*, that is, an abelian Lie group whose identity component is a torus (these are product of a torus and a finite group). If we choose the s given by the corresponding automorphism of the Dynkin diagram, its order is $q \in \{1, 2, 3\}$ and $S = T \times \mathbb{Z}/q\mathbb{Z}$.

As the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is $\mathfrak{g}$, S acts on $\mathfrak{g}$. The weight space decomposition for S is the θ-root decomposition, and the θ-roots are the nonzero characters $\Delta \subseteq$

$\mathfrak{X}(S) = \mathfrak{X}(T) \times \mathbb{Z}/q\mathbb{Z}$. The restriction of the θ-roots to T (except for those restricting to zero) is called the set of (restricted) *real θ-roots*, $\overline{\Delta} = \Delta|_{\mathfrak{X}(T)} \setminus \{0\} \subseteq \mathfrak{X}(T)$. This is a (possibly non-reduced) root system [27, Theorem 3.3.14] and its simple roots are the desired system of simple θ-roots. The extended simple root system giving rise to the Kac diagram is obtained by considering the $\mathbb{Z}/q\mathbb{Z}$-grading of $\mathfrak{g}$ induced by s, taking η to be the highest weight of the representation of the zero-th piece in the first one, and adding $(-\eta, 1)$ to the system of simple roots. See also [31, Section X.5] for a detailed explanation and proof (due to Kac) of the classification in the outer case, using covering Lie algebras of infinite dimension.

We have listed all Kac diagrams in Table 2.1. The labellings correspond to the coefficients n_i. The white vertex is the extended root, meaning that removing it results in the Dynkin diagram for the corresponding simple θ-roots. The *type* refers to the Dynkin type of the simple θ-root system.

Example 7. Consider $\mathfrak{g} = \mathfrak{sl}_6(\mathbb{C})$. Recall the conventions for the roots, the Dynkin diagram and the notation from Example 3. The order two automorphism of the Dynkin diagram interchanges the root α_i with the root α_{6-i} for $i \in \{1, \dots, 5\}$. This can be realised by fixing the symmetric bilinear form with matrix

$$Q = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix},$$

and considering the Lie algebra involution $s : A \mapsto -A^t$, where the transpose is with respect to Q, i.e. a symmetry along the main antidiagonal of the matrix. This involution gives a non-inner 2-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{so}_6(\mathbb{C})$ and $\mathfrak{g}_1 = \mathrm{Sym}_6(\mathbb{C}^n)$. This is the usual decomposition of a matrix in its skew-symmetric and symmetric parts with respect to the bilinear symmetric form Q. Using the previous theory, we can construct a $\mathbb{Z}/m\mathbb{Z}$-grading of the same outer class as s for even values of m. Let us explore the case of $m = 4$.

The maximal torus T can be taken to be:

$$T = \{\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}) : \lambda_i \in \mathbb{C}^*\} = (\mathbb{C}^*)^3.$$

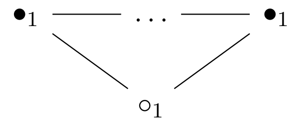
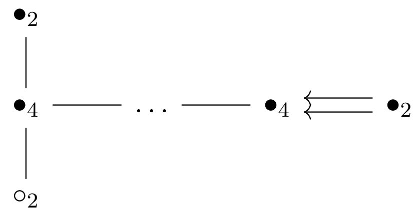
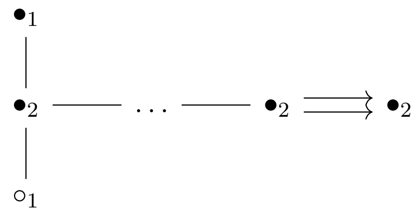
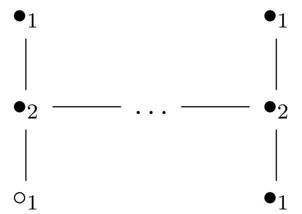
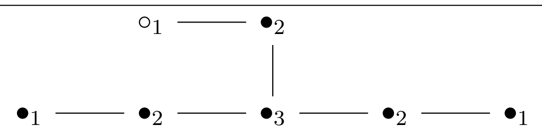
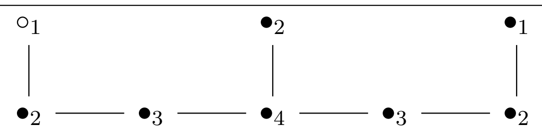
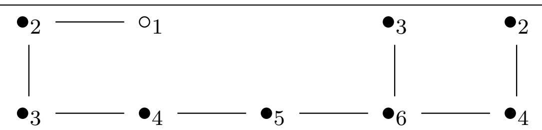
It acts on $\mathfrak{g}$ by conjugation. The cyclic group $\mathbb{Z}/2\mathbb{Z}$ also acts on $\mathfrak{g}$ by the action of s. The quasitorus is then $S = T \times \mathbb{Z}/2\mathbb{Z}$. The characters are $\mathfrak{X}(S) = \mathfrak{X}(T) \times \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}^3 \times \mathbb{Z}/2\mathbb{Z}$.

From Table 2.1 we see that the resulting Kac diagram is

$$\begin{array}{ccc} \bullet_2 & & \\ | & & \\ \bullet_4 & \Longleftarrow & \bullet_2 \\ | & & \\ \circ_2 & & \end{array},$$

where a computation reveals that the root labelled with a 4 can be taken to be the element $\alpha_1 := (-1, 1, 0; 0)$, the other endpoint of the double arrow is $\alpha_0 := (2, 0, 0; 1)$

Table 2.1: Kac diagrams for simple Lie algebras.

Case	**Type**	**Diagram**
$\mathfrak{sl}_n(\mathbb{C})$ (inner)	A_n	
$\mathfrak{sl}_{2n}(\mathbb{C})$ (non-inner)	C_n	
$\mathfrak{sl}_{2n+1}(\mathbb{C})$ (non-inner)	BC_n	$\circ_2 \Rightarrow \bullet_4 - \cdots - \bullet_4 \Rightarrow \bullet_4$
$\mathfrak{so}_{2n+1}(\mathbb{C})$ (inner)	B_n	
$\mathfrak{sp}_{2n}(\mathbb{C})$ (inner)	C_n	$\circ_1 \Rightarrow \bullet_2 - \cdots - \bullet_2 \Leftarrow \bullet_1$
$\mathfrak{so}_{2n}(\mathbb{C})$, $n \geq 2$ (inner)	D_n	
$\mathfrak{so}_{2n}(\mathbb{C})$ (non-inner)	B_{n-1}	$\circ_2 \Leftarrow \bullet_2 - \cdots - \bullet_2 \Rightarrow \bullet_2$
$\mathfrak{so}_8(\mathbb{C})$ (non-inner II)	G_2	$\bullet_3 \Rrightarrow \bullet_6 - \circ_3$
$\mathfrak{e}_6$ (inner)	E_6	
$\mathfrak{e}_6$ (non-inner)	F_4	$\circ_2 - \bullet_4 - \bullet_6 \Leftarrow \bullet_4 - \bullet_2$
$\mathfrak{e}_7$ (inner)	E_7	
$\mathfrak{e}_8$ (inner)	E_8	
$\mathfrak{f}_4$ (inner)	F_4	$\circ_1 - \bullet_2 - \bullet_3 \Rightarrow \bullet_4 - \bullet_2$
$\mathfrak{g}_2$ (inner)	G_2	$\circ_1 - \bullet_2 \Rrightarrow \bullet_3$

and the remaining simple roots can be taken to be $\alpha_2 := (0,-1,1;0)$ and $\alpha_3 := (0,-1,-1;0)$. We verify that indeed $2\alpha_0 + 4\alpha_1 + 2\alpha_2 + 2\alpha_3 = (0,0,0;0)$. Let us examine the $\mathbb{Z}/4\mathbb{Z}$-grading corresponding to the assignation of the label 1 to α_1 and 0 to the remaining vertices. Computing the labels of each θ-root via linear combinations of the elements in the Kac diagram, we can describe the grading as follows. Given a matrix $A = A_0 + A_1 \in \mathfrak{sl}_6(\mathbb{C})$, where $A_i \in \mathfrak{g}_i$ (recall this is the grading given by s, in other words, A_0 skew-symmetric and A_1 symmetric with respect to Q), the following two diagrams give which parts of A_0 and A_1, respectively, belong on the i-th piece of the $\mathbb{Z}/4\mathbb{Z}$ grading (the entries labelled with a $j \neq i$ will be zero in an element of $\mathfrak{g}_i$):

$$\begin{pmatrix} 0 & 3 & 3 & 3 & 3 & \\ 1 & 0 & 0 & 0 & & 3 \\ 1 & 0 & 0 & & 0 & 3 \\ 1 & 0 & & 0 & 0 & 3 \\ 1 & & 0 & 0 & 0 & 3 \\ & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 2 & 2 & 2 & 1 \\ 3 & 2 & 2 & 2 & 2 & 1 \\ 3 & 2 & 2 & 2 & 2 & 1 \\ 3 & 2 & 2 & 2 & 2 & 1 \\ 0 & 3 & 3 & 3 & 3 & 2 \end{pmatrix}.$$

The grading can be described abstractly as follows: if $\mathfrak{sl}_6(\mathbb{C}) = \mathfrak{sl}(V)$ for V a 6-dimensional complex vector space, we fix a splitting $V = V_{-1} \oplus V_1$ with $\dim V_{-1} = 2$, $\dim V_1 = 4$. We fix a non-degenerate bilinear form f on V that splits as a symmetric bilinear form on V_1 and a skew-symmetric bilinear form on V_{-1}. Then the zero-th piece of the grading are the traceless elements of $\mathrm{End}(V_{-1}) \oplus \mathrm{End}(V_1)$ skew-symmetric with respect to f (that is, the resulting algebra is $\mathfrak{so}(V_1) \oplus \mathfrak{sp}(V_{-1})$), the first piece are the $A \in \mathrm{Hom}(V_1, V_{-1}) \oplus \mathrm{Hom}(V_{-1}, V_1)$ with $f(Ax,y) + if(x,Ay) = 0$, the second piece are the traceless elements of $\mathrm{End}(V_{-1}) \oplus \mathrm{End}(V_1)$ which are symmetric with respect to f and the third piece are the $A \in \mathrm{Hom}(V_1, V_{-1}) \oplus \mathrm{Hom}(V_{-1}, V_1)$ with $f(Ax,y) - if(x,Ay) = 0$. The corresponding automorphism of the group $\mathrm{SL}_6(\mathbb{C})$ is given by $g \mapsto (g^t)^{-1}$, where the transpose is with respect to f.

In general, Vinberg [50] gives a description for non-inner $\mathbb{Z}/m\mathbb{Z}$-gradings θ of $\mathfrak{sl}_n(\mathbb{C})$ as follows. One has that θ is conjugate to $g \mapsto (g^t)^{-1}$, where t is the transpose with respect to some non-degenerate bilinear form f on $\mathbb{C}^n$. Let $a \in \mathrm{SL}_n(\mathbb{C})$ be defined by $f(x,y) = f(y,ax)$ (equivalently, by $\theta^2(g) = aga^{-1}$). Split $\mathbb{C}^n = \bigoplus_{\lambda \in S} V_\lambda$ in eigenspaces for a, which can be chosen so that its eigenvalues S are either $\frac{m}{2}$-th roots of unity or $\frac{m}{2}$-th roots of -1.

As m is even, let ζ' be a primitive $\frac{m}{2}$-th root of unity and ζ a primitive m-th root of unity. Then, the i-th piece of the grading given by θ consists of the traceless linear operators $A \in \mathfrak{sl}_n(\mathbb{C})$ such that $AV_\lambda \subseteq V_{(\zeta')^i\lambda}$ and $f(Ax,y) + \zeta^i f(x,Ay) = 0$.

2.2.2. Cyclic gradings associated to $\mathbb{Z}$-gradings

Our main objects of study in the following chapters will be Higgs bundles associated to cyclic gradings, focusing on the case of Example 5. In order to exploit the theory of prehomogeneous vector spaces and $\mathbb{Z}$-gradings from Section 2.1 as well, we are going to see how $\mathbb{Z}$-gradings can be related to $\mathbb{Z}/m\mathbb{Z}$-gradings.

Definition 10. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a $\mathbb{Z}$-grading. We define the **associated $\mathbb{Z}/m\mathbb{Z}$-grading** as follows: by Proposition 1, we have a homomorphism $\gamma : \mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g})$. Precomposing with the inclusion $\mu_m \hookrightarrow \mathbb{C}^*$ gives a homomorphism $\gamma' : \mu_m \to \mathrm{Aut}(\mathfrak{g})$ which by Proposition 4 is a $\mathbb{Z}/m\mathbb{Z}$-grading.

The pieces of the two gradings can be easily related.

Proposition 5. *Let $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ be a $\mathbb{Z}$-graded complex semisimple Lie algebra. Then the associated $\mathbb{Z}/m\mathbb{Z}$-grading, $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}/m\mathbb{Z}} \bar{\mathfrak{g}}_j$, is given by $\bar{\mathfrak{g}}_j := \bigoplus_{k\equiv j \mod m} \mathfrak{g}_k$.*

Proof. Consider $\theta \in \mathrm{Aut}_m(\mathfrak{g})$ giving the associated cyclic grading, obtained from the definition above as the image of a primitive root of unity $\zeta \in \mu_m$. We then have for $X \in \mathfrak{g}_k$ that $\theta(X) = \zeta^k X$. Thus $\mathfrak{g}_k \subseteq \bar{\mathfrak{g}}_k$. □

Remark 5. Given a $\mathbb{Z}$-graded complex semisimple Lie algebra $\mathfrak{g}$, and a positive integer $m \in \mathbb{N}$ such that $\mathfrak{g}_i = 0$ if $|i| \geq m$, the associated cyclic grading for this m has $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$, and $\bar{\mathfrak{g}}_i = \mathfrak{g}_i \oplus \mathfrak{g}_{i-m}$ for $i \in \{1, \ldots, m-1\}$. Thus, the corresponding prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ (as well as $(G_0, \mathfrak{g}_{1-m})$) and the Vinberg θ-pair $(G_0, \mathfrak{g}_1 \oplus \mathfrak{g}_{1-m})$ have the same group G_0 and we will use both for some results in the following chapters.

Thus, we introduce the following definition:

Definition 11. A $\mathbb{Z}/m\mathbb{Z}$-grading of a complex semisimple Lie algebra $\mathfrak{g}$ is said to be **special** if it is associated to a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ such that $\mathfrak{g}_i = 0$ whenever $|i| \geq m$. This $\mathbb{Z}$-grading is called a **special associated $\mathbb{Z}$-grading**.

We list two important examples of special gradings.

Example 8. Let $G^{\mathbb{R}} \subseteq G$ be a real form and suppose that the resulting Vinberg θ-pair $(H, \mathfrak{m})$ from Example 4 arises from a special $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, that is, we have $\mathfrak{h} = \mathfrak{g}_0$ and $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. These real forms are called of **hermitian type**, as they arise naturally in the classification of hermitian symmetric spaces.

Example 9. The Vinberg θ-pair in Example 5 is special, associated to the prehomogeneous vector space from Example 2.

Using the classification of cyclic gradings given at the end of the previous section, we can discuss the cases in which such a grading comes from a $\mathbb{Z}$-grading and, as it will be a crucial hypothesis for Chapter 4, those in which it is special, so that both gradings have the same group G_0. We work in the inner case.

Applying the same reductions and notation as in previous section, let $\mathfrak{g}$ be simple and $\theta \in \mathrm{Int}_m(\mathfrak{g})$ be an order m semisimple inner automorphism, which by the already established classification we can take to be $\theta = \exp(2\pi i x)$ for $x \in \mathfrak{t}$ coming from a labelling $\{p_0, \ldots, p_r\}$ of the affine Dynkin diagram with $\sum_{i=0}^r n_i p_i = m$. If we obtain an associated $\mathbb{Z}$-grading for this cyclic grading, we have an associated $\mathbb{Z}$-grading for any inner cyclic grading, because we can relate it to this one via (inner) automorphism.

Notice that this labelling restricts to a labelling $\{p_1, \ldots, p_r\}$ of the Dynkin diagram, which gives a $\mathbb{Z}$-grading. Let $\alpha \in \mathfrak{t}^*$ be a root, and write

$$\alpha = \sum_{i=0}^{r} k_i \alpha_i, \qquad 0 \leq k_i \leq n_i.$$

Then, if we denote by $\mathfrak{g}_j$ the pieces of this $\mathbb{Z}$-grading and by $\bar{\mathfrak{g}}_j$ the pieces of the starting $\mathbb{Z}/m\mathbb{Z}$-grading, we have that $\mathfrak{g}_\alpha \subseteq \bar{\mathfrak{g}}_p$, with $p = \sum_{i=0}^r k_i p_i$. On the other hand, using that $\alpha_0 = -\sum_{i=1}^r n_i \alpha_i$, we get that

$$\alpha = \sum_{i=1}^r (k_i - n_i k_0)\alpha_i,$$

so that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{p'}$ for $p' = \sum_{i=1}^r (k_i - n_i k_0) p_i$. However, using that $m = \sum_{i=0}^r n_i p_i$, we have

$$p' = \sum_{i=1}^r (k_i - n_i k_0) p_i = \sum_{i=1}^r k_i p_i - k_0 \sum_{i=1}^r n_i p_i = \sum_{i=1}^r k_i p_i - k_0(m - p_0) =$$

$$= \sum_{i=0}^r k_i p_i - k_0 m = p - k_0 m.$$

This means that both labels are congruent mod m, so that the $\mathbb{Z}$-grading induces the $\mathbb{Z}/m\mathbb{Z}$ grading (in both cases $\mathfrak{t}$ is in the 0-th piece). Thus:

Proposition 6. *Any inner $\mathbb{Z}/m\mathbb{Z}$-grading of a semisimple Lie algebra $\mathfrak{g}$ is associated to a $\mathbb{Z}$-grading.*

However, the lift obtained above will **not** be in general special, which is desirable so that, for example, the group G_0 agrees in both the Vinberg $\mathbb{C}^*$-pair and the Vinberg θ-pair. For this, we need that the index $p' = p - k_0 m$ of any root α in the $\mathbb{Z}$-grading verifies $p' \in \{1-m, \dots, m-1\}$. We have, since $0 \le k_i \le n_i$, that $p = \sum_{i=0}^r k_i p_i$ satisfies $0 \le p \le \sum_{i=0}^r n_i p_i = m$, so that $-k_0 m \le p' \le (1-k_0)m$. As $k_0 \in \{0,1\}$, the only problematic cases are:

- $p' = -m$, in which case $k_0 = 1$, $p = 0$ and $k_i = 0$ for all $i \in \{1, \dots, r\}$, which implies that $p_0 = 0$ (and, conversely, if $p_0 = 0$ the lowest root space sits in the $-m$-th piece of the $\mathbb{Z}$-grading).

- $p' = m$, in which case $k_0 = 0$ and $k_i = n_i$ for all $i \in \{1, \dots, r\}$, which means, from the fact that $m = p' = \sum_{i=0}^r k_i p_i = \sum_{i=1}^r n_i p_i$, that $p_0 = 0$ (and, conversely, if $p_0 = 0$ the highest root space sits in the m-th piece of the $\mathbb{Z}$-grading).

So, the resulting $\mathbb{Z}$-grading is special if and only if the label p_0 of the lowest root is nonzero. Recall from Theorem 4 that acting on the labels by Γ gives the same cyclic grading up to conjugation by inner automorphism, and acting by the full automorphism group $\mathrm{Aut}(\Pi')$ of the affine Dynkin diagram gives the same cyclic grading, this time up to (possibly non-inner) automorphism. If we can take a nonzero label to α_0 via this action, we obtain a lifting $\mathbb{Z}$-grading of the desired form, which can be then translated back to the cyclic grading of the starting labelling via the automorphism.

Thus, we have proven:

Proposition 7. *Suppose that an inner $\mathbb{Z}/m\mathbb{Z}$-grading of a semisimple Lie algebra $\mathfrak{g}$ is given by a labelling $\{p_0, \dots, p_r\}$ of the affine Dynkin diagram corresponding to the affine simple roots $\Pi' = \{\alpha_0, \dots, \alpha_r\}$.*

If there exists a non-zero label p_i such that $\alpha_i \in \mathrm{Aut}(\Pi') \cdot \alpha_0$, then the grading is special.

Example 10. The cyclic grading of Example 5, whose labelling was showcased in Example 6, is special as we had already noticed. As the label in the extended root α_0 is 1, the lifting $\mathbb{Z}$-grading is the one given by restricting the labelling to the Dynkin diagram, which is precisely the labelling of Example 3 giving the grading of Example 2, as we observed.

Remark 6. Recall that the subgroup $\Gamma \trianglelefteq \mathrm{Aut}(\Pi')$ acts simply transitively on the set of affine simple roots α_i such that $n_i = 1$, which includes α_0. Thus, if there is a root α_i with $n_i = 1$ such that $p_i > 0$, there exists a special associated $\mathbb{Z}$-grading.

This remark has as a consequence the following interesting example:

Example 11. Any inner cyclic grading of $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ is special. This is because in type A_{n-1} we have $n_i = 1$ for all $i \in \{0, \dots, n-1\}$ or, in other words, Γ acts transitively on the affine Dynkin diagram, as can be noticed in Example 6.

Example 12. Let us showcase an example where $\mathrm{Aut}(\Pi')$ does not act transitively, so that Proposition 7 fails. We can look in type B_n, which corresponds to the Lie algebra $\mathfrak{so}_{2n+1}(\mathbb{C})$. We can work with the following description. Let

$$Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

This is a symmetric bilinear form, so that $\mathfrak{so}_{2n+1} = \{X \in \mathrm{Mat}_{2n+1}(\mathbb{C}) : QX + X^t Q = 0\}$. These are matrices of the form:

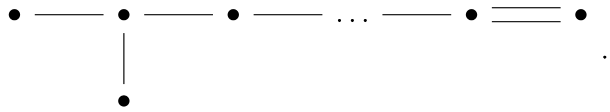
$$X = \begin{pmatrix} 0 & b^t & -a^t \\ a & A & B \\ -b & C & -A^t \end{pmatrix},$$

where $B^t = -B$ and $C^t = -C$. We can take the Cartan subalgebra $\mathfrak{t}$ of diagonal matrices in $\mathfrak{so}_{2n+1}(\mathbb{C})$. For $i \in \{1, \dots, n\}$ Let $e_i \in \mathfrak{t}^*$ be the linear form that reads the i-th element of the diagonal of A as in the previous form (that is, the $(i+1)$-th element of the diagonal in X). The simple roots can be taken to be $\Pi = \{\alpha_1, \dots, \alpha_n\}$ with $\alpha_i = e_{i+1} - e_i$ for $i \in \{1, \dots, n-1\}$ and $\alpha_n = -e_n$. The roots are $\pm e_i \pm e_j$ for $i \neq j$ and $\pm e_i$. Thus, the only sums of simple roots which are roots are $\alpha_i + \alpha_{i+1}$ and the Dynkin diagram is:

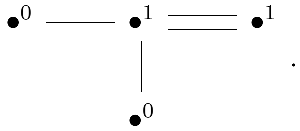
$$\bullet \;\text{——}\; \bullet \;\text{——}\; \bullet \;\text{——}\; \dots \;\text{——}\; \bullet \;=\!=\; \bullet\,.$$

The last edge is double because the angle between α_{n-1} and α_n is different (this will not be relevant for this example; similarly, we have omitted an arrow pointing from

α_{n-1} to α_n indicating that the former is longer than the latter). The highest root is $-e_2-e_1$, which corresponds to $n_i = 2$ for $i \geq 2$, $n_1 = 1$. Thus $\alpha_0 = e_2+e_1$, and then $\alpha_i + \alpha_0$ is only a root for $i = 2$. The affine Dynkin diagram of type B_n is then:



Its automorphism group (for $n \geq 3$) is $\mathrm{Aut}(\Pi') = \mathbb{Z}/2\mathbb{Z}$, acting by swapping α_0 and α_1. It equals Γ since $\mathrm{Aut}(\Pi) = \{1\}$, and acts transitively on the roots α_i with $n_i = 1$, as expected. Then, if $p_0 = p_1 = 0$, we cannot apply Proposition 7. For example, consider in the case $n = 3$ the labelling



This gives a cyclic grading for $m = 2 \cdot 1 + 2 \cdot 1 = 4$, summarized in the following diagram:

$$\begin{pmatrix} & 2 & 2 & 1 & 2 & 2 & 3 \\ 2 & 0 & 0 & 3 & & 0 & 1 \\ 2 & 0 & 0 & 3 & 0 & & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 & \\ 2 & & 0 & 3 & 0 & 0 & 1 \\ 2 & 0 & & 3 & 0 & 0 & 1 \\ 1 & 3 & 3 & & 3 & 3 & 0 \end{pmatrix},$$

interpreted in the following way: $\mathfrak{g}_i$ are the elements of $\mathfrak{so}_7\mathbb{C}$ whose entries not labelled with i in the diagram above are zero (the blank spaces are always zero in $\mathfrak{so}_7(\mathbb{C})$). The associated $\mathbb{Z}$-grading given by restriction of the labelling to the Dynkin diagram is given by the following diagram:

$$\begin{pmatrix} & 2 & 2 & 1 & -2 & -2 & -1 \\ -2 & 0 & 0 & -1 & & -4 & -3 \\ -2 & 0 & 0 & -1 & -4 & & -3 \\ -1 & 1 & 1 & 0 & -3 & -3 & \\ 2 & & 4 & 3 & 0 & 0 & 1 \\ 2 & 4 & & 3 & 0 & 0 & 1 \\ 1 & 3 & 3 & & -1 & -1 & 0 \end{pmatrix},$$

interpreted in the same way. Both diagrams are the same mod 4, so indeed this $\mathbb{Z}$-grading induces the previous $\mathbb{Z}/4\mathbb{Z}$-grading, but we see that this $\mathbb{Z}$-grading has a smaller 0-th piece than the $\mathbb{Z}/4\mathbb{Z}$-grading.

CHAPTER 3

Higgs bundles and their moduli spaces

3.1. Basics on moduli spaces of Higgs bundles

Higgs bundles were originally introduced by Nigel Hitchin in his seminal paper [34], as these objects appeared naturally in his study of self-duality equations on a Riemann surface. From then on, the theory has evolved in multiple directions and connections with different branches of mathematics have emerged. In this section we introduce the basics of Higgs bundles with the level of generality that we will need, while illustrating some of the original motivation with examples throughout.

3.1.1. Main definitions and stability

During this section, we fix a compact Riemann surface X of genus $g \geq 2$ and we denote by $K_X := T^*X$ its canonical line bundle. We start with the definition of a Higgs bundle associated to a representation [5, Definition 4.1].

Definition 12. Let G be a complex reductive Lie group and $\rho : G \to GL(V)$ a holomorphic representation. Let L be a line bundle over X. An **L-twisted (G, V)-Higgs pair** is a pair (E, φ), where E is a holomorphic principal G-bundle on X, and $\varphi \in H^0(X, E(V) \otimes L)$, where $E(V) := E \times_G V$ is the associated vector bundle to E via ρ. A **(G, V)-Higgs pair** is a K_X-twisted (G, V)-Higgs pair.

The section φ is usually called *Higgs field*. By default we will work with (G, V)-Higgs pairs (that is, K_X-twisted), and in the few cases where the twisting line bundle is different from K_X we will explicitely state it. The definition in this level of generality specializes to give some of the most studied examples.

Example 13. Denote by $\mathfrak{g}$ the Lie algebra of G. When $V = \mathfrak{g}$, and $\rho = \mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation, the resulting pair is known as a **G-Higgs bundle** and corresponds to the objects originally studied by Hitchin in [32, 34]. Besides the results about their moduli space that we will see later, we can already explain a strong reason that motivates the study of these objects. Consider the moduli space N of stable principal G-bundles on X. By differential geometric means it can be seen [8] that its tangent space at the isomorphism class of a principal bundle E is $T_E N = H^1(X, E(\mathfrak{g}))$. Using Serre duality, we get that the cotangent space is $T_E^* N = H^1(X, E(\mathfrak{g}))^* = H^0(X, E(\mathfrak{g}) \otimes K_X)$. Thus, the cotangent bundle T^*N consists of G-Higgs bundles. This fact can be used to endow the space of G-Higgs bundles with a symplectic structure.

Example 14. More concretely, we can take G to be a Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ in the previous example, which allows to see the corresponding G-Higgs bundles as holomorphic vector bundles with extra structure and an associated section. For example, if $G = GL_n(\mathbb{C})$ then a G-bundle E can be seen as a holomorphic vector bundle of rank n (by taking $E \times \mathbb{C}^n$ the associated vector bundle to the standard representation of $\mathrm{GL}_n(\mathbb{C})$, so that E is the frame bundle to the resulting vector bundle). Denoting such bundle by E as well, the Higgs field becomes a section $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$, since $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ are the endomorphisms of $\mathbb{C}^n$.

Choosing other complex Lie subgroups of $\mathrm{GL}_n(\mathbb{C})$ implies adding extra structure to the vector bundle E and imposing extra conditions in the section φ. For example, if $G = \mathrm{SL}_n(\mathbb{C})$ are automorphisms with determinant 1, then $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ are traceless endomorphisms, and thus the resulting vector bundle has $\det E = \mathcal{O}$, and the Higgs field has $\operatorname{tr} \varphi = 0$. Another example is $G = \mathrm{Sp}_{2n}(\mathbb{C})$, for which the vector bundle E is endowed with a symplectic form $\Omega : E \otimes E \to \mathcal{O}$, and the Higgs field verifies $\Omega(\varphi v, w) + \Omega(v, \varphi w) = 0$.

Example 15. Let G be complex reductive and $G^{\mathbb{R}} \subseteq G$ a real form, and recall the associated Vinberg θ-pair $(H, \mathfrak{m})$ from Example 4. Higgs bundles associated to this pair are called **$G^{\mathbb{R}}$-Higgs bundles**. This is the natural extension of the theory of Higgs bundles to real reductive groups, studied in [21] (see also the survey [19] and references therein). We will give some motivation for this definition once we define the moduli space at the end of the section.

Example 16. As in Example 14, we can see Higgs bundles for real Lie groups in terms of vector bundles when these groups occur as subgroups of $\mathrm{GL}_n(\mathbb{C})$. For example, consider $G = \mathrm{GL}_n(\mathbb{C})$ and the real form given by the antiholomorphic involution $\sigma : A \mapsto I_{p,q}(\bar{A}^T)^{-1} I_{p,q}$, where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ for $p + q = n$. This real form is the *unitary group for a hermitian form of signature* (p, q) and is denoted by $\mathrm{U}(p, q)$. It consists of the automorphisms A that preserve the hermitian form of signature (p, q) given by $I_{p,q}$. A maximal compact is $H^{\mathbb{R}} = \mathrm{U}(p) \times \mathrm{U}(q)$, which embeds in $\mathrm{U}(p, q)$ diagonally. Thus, $H = \mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$, so $\mathfrak{h} = \mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_q(\mathbb{C})$, i.e. endomorphisms given by a $p \times p$ and a $q \times q$ block on the diagonal, and then $\mathfrak{m}$ is precisely off-diagonal endomorphisms.

Consequently, in this case a principal H-bundle is a rank n vector bundle $E = V \oplus W$, where V and W are subbundles of ranks p and q respectively, and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$ must satisfy the extra property $\varphi(V) \subseteq W \otimes K_X$, $\varphi(W) \subseteq V \otimes K_X$ (so that the condition of being off-diagonal is met).

Having defined what Higgs bundles are, we want to consider *moduli spaces*, that is, the sets of their isomorphism classes, where we understand two (G, V)-Higgs pairs

(E, φ) and (E', φ') to be isomorphic if there is a G-bundle isomorphism $f : E \to E'$ carrying φ to φ'. However, the resulting set of isomorphism classes does not have good geometric properties (for example, it is not Hausdorff). The solution, given by Geometric Invariant Theory, is to restrict the attention to certain subset of the set of all isomorphism classes such that the resulting space has the structure of a manifold or an algebraic variety. For this we need to define notions of *stability* for (G, V)-Higgs pairs. We follow [21, Definition 2.9] and [5, Definition 4.11]. First, we need the notion of *reduction* for Higgs pairs.

Definition 13. Let G be a complex reductive Lie group, $\hat{G} \leq G$ a Lie subgroup, and E a principal G-bundle. A **reduction of structure group** of E to $\hat{G}$ is a holomorphic section $\sigma \in H^0(X, E(G/\hat{G}))$, where $E(G/\hat{G}) = E \times_G G/\hat{G}$.

Remark 7. The natural map $E \to E(G/\hat{G})$ has $\hat{G}$-bundle structure. From the previous definition, a reduction of the structure group is a map $\sigma : X \to E(G/\hat{G})$. This means that we can pull back the $\hat{G}$-bundle E on $E(G/\hat{G})$ to get $E_\sigma := \sigma^* E$, a $\hat{G}$-bundle on X, whence the name *reduction of structure group*. Moreover, there is a canonical isomorphism $E_\sigma \times_{\hat{G}} G \simeq E$. The map $E_\sigma = \sigma^* E \to E$ induced by the pullback is injective and gives a holomorphic subvariety $E_\sigma \subseteq E$.

Definition 14. Let G be a complex reductive Lie group and $\rho : G \to \mathrm{GL}(V)$ a holomorphic representation. Let $\hat{G} \leq G$ be a Lie subgroup and $\hat{V} \subseteq V$ be a $\rho(\hat{G})$-invariant vector subspace. A (G, V)-Higgs pair (E, φ) **reduces to a** $(\hat{G}, \hat{V})$**-Higgs pair** if there is a reduction of the structure group σ of E to $\hat{G}$ such that $\varphi \in H^0(X, E_\sigma(\hat{V}) \otimes K_X) \subseteq H^0(X, E(V) \otimes K_X)$.

For the definition we use that, since $\hat{V}$ is $\rho(\hat{G})$-invariant, we have a restriction $\hat{\rho} : \hat{G} \to \mathrm{GL}(\hat{V})$, as well as the fact that, since $E_\sigma \subseteq E$ and $\hat{V} \subseteq V$, there is a well defined inclusion of vector bundles $E_\sigma(\hat{V}) \subseteq E(V)$.

Now we set up the definition for stability. Start by fixing a maximal compact subgroup $K \leq G$ (not to be confused with the canonical line bundle of X, which we have intentionally denoted as K_X). Let $\mathfrak{k}$ be its Lie algebra, a real subalgebra of $\mathfrak{g}$. Define for $s \in i\mathfrak{k}$ the spaces

$$V_s^0 = \{v \in V : \forall t \in \mathbb{R}, \rho(e^{ts})(v) = v\}, \quad V_s = \{v \in V : \rho(e^{ts})(v) \text{ is bounded as } t \to \infty\}.$$

and the subgroups

$$L_s = \{g \in G : \mathrm{Ad}(g)(s) = s\}, \quad P_s = \{g \in G : e^{ts} g e^{-ts} \text{ is bounded as } t \to \infty\}.$$

These subgroups of G correspond to the lie algebras $\mathfrak{g}_s^0$ and $\mathfrak{g}_s$ defined for the adjoint representation. We also define a character $\chi_s : \mathfrak{g}_s \to \mathbb{C}$ given by $\chi_s(x) = B(s, x)$, where B is the Killing form on $\mathfrak{g}$.

The subgroup L_s acts on V_s^0 via ρ: if $g \in L_s$, then $\mathrm{Ad}(g)(s) = s$ so that $g e^{ts} g^{-1} = e^{ts}$. Thus, if $v \in V_s^0$, we get $\rho(e^t s)(\rho(g)(v)) = \rho(e^{ts} g)(v) = \rho(g e^{ts})(v) = \rho(g)(\rho(e^{ts})(v)) = \rho(g)(v)$, so that $\rho(g)(v) \in V_s^0$. Similarly, P_s acts on V_s: if $g \in P_s$ then $\rho(e^{ts})(\rho(g)(v)) = \rho(e^{ts} g e^{-ts})(\rho(e^{ts})(v))$, which is bounded as $t \to \infty$ since both parts in that expression are.

Let E be a G-bundle and let $\sigma \in H^0(X, E(G/P_s))$ be a reduction of E to P_s. If a multiple $q\chi_s$ for some $q \in \mathbb{N}$ lifts to a character $\tilde{\chi}_s : P_s \to \mathbb{C}^*$, we define the **degree of the reduction** as

$$\deg E(\sigma, s) := \frac{1}{q} \deg(E_\sigma \times_{\tilde{\chi}_s} \mathbb{C}^*).$$

It is also possible to define this quantity when no multiple of the character lifts to the group, using differential geometric techniques, as follows: given a reduction σ of E to P_s, there is a further reduction σ' to $K_s := K \cap L_s$, the maximal compact of L_s. Take a connection A on $E_{\sigma'}$ and consider its curvature $F_A \in \Omega^2(X, E_{\sigma'}(\mathfrak{k}_s))$. We have that $\chi_s(F_A) \in \Omega^2(X, i\mathbb{R})$, and the degree is defined as

$$\deg E(\sigma, s) := \frac{i}{2\pi} \int_X \chi_s(F_A).$$

We refer to [5, Section 4.2] for more details on this.

Finally, let $\mathfrak{z}$ be the center of $\mathfrak{k}$, so that $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_{ss}$, where $\mathfrak{k}_{ss}$ is the semisimple part. Given the representation $\rho : G \to \mathrm{GL}(V)$, let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be its differential, and define

$$\mathfrak{k}_\rho := \mathfrak{k}_{ss} \oplus \ker(d\rho|_{\mathfrak{k}})^\perp,$$

where the orthogonal is taken in $\mathfrak{k}$ with respect to the Killing form.

The last ingredient we need is required in order to deal with the possibility of G being non-connected, which we do not exclude in our definition of reductive. This follows [22]. Denote by $G_0 \leq G$ the connected component of the identity, meaning that we have an extension $1 \to G_0 \to G \to \Gamma \to 1$ where we assume that $\Gamma = \pi_0(G)$ is finite. By results of [22], there exists an action $\theta : \Gamma \to \mathrm{Aut}(G_0)$ and an homomorphism $c : \Gamma \times \Gamma \to Z(G_0)$ (which, in terms of group cohomology, is moreover a cocycle with respect to the action θ), such that $G \simeq G_0 \times_{(\theta,c)} \Gamma$, the latter subscript meaning that the group operation is given by $(g_1, \gamma_1) \cdot (g_2, \gamma_2) = (g_1\theta(\gamma_1)(g_2)c(\gamma_1, \gamma_2), \gamma_1\gamma_2)$. The maximal compact K can also be taken to be Γ-invariant. Thus it makes sense to consider the fixed points $\mathfrak{z}^\Gamma \subseteq \mathfrak{z}$ and $\mathfrak{k}^\Gamma \subseteq \mathfrak{k}$.

We can now define stability.

Definition 15. Fix a parameter $\alpha \in i\mathfrak{z}^\Gamma$. A (G, V)-Higgs pair (E, φ) is:

- α-**semistable**, if for any element $s \in i\mathfrak{k}^\Gamma$ and reduction $\sigma \in H^0(X, E(G/P_s))$ such that $\varphi \in H^0(E_\sigma(V_s) \otimes K_X)$, we have $\deg E(\sigma, s) \geq B(\alpha, s)$.

- α-**stable**, if it is α-semistable and, for any element $s \in i\mathfrak{k}_\rho^\Gamma$ and reduction $\sigma \in H^0(X, E(G/P_s))$ such that $\varphi \in H^0(E_\sigma(V_s) \otimes K_X)$, we have $\deg E(\sigma, s) > B(\alpha, s)$.

- α-**polystable**, if it is α-semistable and, for the $s \in i\mathfrak{k}^\Gamma$ and $\sigma \in H^0(X, E(G/P_s))$ such that $\varphi \in H^0(E_\sigma(V_s) \otimes K_X)$ and we have $\deg E(\sigma, s) = B(\alpha, s)$, there exists a reduction $\sigma' \in H^0(E_\sigma(P_s/L_s))$ of E_σ to L_s such that $\varphi \in H^0(E_{\sigma'}(V_s^0) \otimes K_X)$.

With this notion, we can now define the **moduli space of α-polystable (G, V)-Higgs pairs** over X as the set of isomorphisms classes of α-polystable (G, V)-Higgs pairs. We denote it by $\mathcal{M}^\alpha(G, V)$. A construction as a geometric space via Geometric Invariant Theory is given by Schmitt in [46]. When $\alpha = 0$, we simply refer to *semistable, polystable* and *stable* Higgs pairs, and denote the moduli space of polystable bundles as $\mathcal{M}(G, V)$. All the previous definitions of stability are identical for L-twisted (G, V)-Higgs pairs for an arbitrary line bundle L over X, simply replacing K_X with L when appropriate. The resulting moduli space of α-polystable L-twisted (G, V)-Higgs pairs will be denoted by $\mathcal{M}^\alpha_L(G, V)$.

Now we will give more details on the moduli spaces for the classical examples explained above, and some extra reasons for their relevance.

Example 17. For G-Higgs bundles defined in Example 13 and $\alpha = 0$, the *moduli space of (polystable) G-Higgs bundles*, denoted by $\mathcal{M}(G)$, is recovered. The original Geometric Invariant Theory construction of this case was done by Simpson [47]. A very important property of this space is the **nonabelian Hodge correspondence**, a consequence of theorems of Corlette [15], Donaldson [18] and the Hitchin–Kobayashi correspondence by Hitchin [34] and Simpson [47, 49]. This correspondence states, for semisimple G, that $\mathcal{M}(G)$ is homeomorphic to the *G-character variety of the fundamental group* $\mathcal{R}^+(G) = \mathrm{Hom}^+(\pi_1(X), G)/G$ of completely reducible representations of the fundamental group of X in G up to conjugation by G, and the subset of stable G-Higgs bundles $\mathcal{M}^s(G)$ is homeomorphic to $\mathcal{R}^*(G) = \mathrm{Hom}^*(\pi_1(X), G)/G$, the subset of $\mathcal{R}^+(G)$ corresponding to irreducible representations. For arbitrary reductive G (not necessarily semisimple), the correspondence still works after possibly replacing $\pi_1(X)$ with its universal central extension Γ.

The dimension of the smooth part of the resulting $\mathcal{M}(G)$ can be studied using the deformation theory on G-Higgs bundles, see for example Biswas and Ramanan [7]. For example, for G semisimple, the resulting dimension is $\dim \mathcal{M}(G) = 2\dim(G)(g-1)$.

Example 18. The particular case of $G \subseteq \mathrm{GL}_n(\mathbb{C})$ described in Example 14 has the advantage that it can be studied in terms of vector bundles, so that a G-Higgs bundle becomes a pair (E, φ) with E a holomorphic vector bundle of rank n on X, and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$, with some extra structure on E and extra conditions on φ depending on the group G. In this setting, the stability conditions become much simpler: we define the **slope** of a vector bundle as $\mu(E) := \frac{\deg E}{\operatorname{rank} E}$, and then (E, φ) is semistable if and only if every proper nonzero vector subbundle $F \subset E$ with $\varphi(F) \subseteq F \otimes K_X$ verifies $\mu(F) \leq \mu(E)$, stable if and only if such subbundles verify $\mu(F) < \mu(E)$, and polystable if E factors as a direct sum of φ-invariant subbundles, all of them stable and with the same slope. The moduli space in the case $G = \mathrm{GL}_n(\mathbb{C})$ was constructed initially by Nitsure [41], resulting in a quasi-projective variety whose smooth locus is the stable subset and has dimension $2 + 2n^2(g-1)$.

Example 19. For the real forms $G^\mathbb{R} \subseteq G$ explained in Examples 15 and 16, the resulting $\mathcal{M}(G^\mathbb{R})$ for $\alpha = 0$ is also very relevant, as it allows to generalize [21] the nonabelian Hodge correspondence to real reductive Lie groups. The correspondence works in the same way, giving a homeomorphism between $\mathcal{M}(G^\mathbb{R})$ and $\mathcal{R}(G^\mathbb{R})$. The particular case of $G^\mathbb{R} = \mathrm{U}(n)$, where $H^\mathbb{R} = G^\mathbb{R}$ and thus $H = G = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{m} = 0$

(so that we always have $\varphi = 0$), establishes a correspondence between (poly)stable holomorphic vector bundles over X and (completely reducible/)irreducible representations of the universal central extension Γ of $\pi_1(X)$ in the unitary group $\mathrm{U}(n)$. This is a classical theorem of Narasimhan and Seshadri [40].

The resulting moduli spaces for certain real forms have been studied in depth. For example, the case of $\mathrm{U}(p,q)$ in [10], or the case of $\mathrm{GL}_n(\mathbb{R})$ in [11].

In following chapters we will need to make use of the Hitchin–Kobayashi correspondence mentioned in Example 17 in the case of Higgs pairs associated to a Vinberg θ-pair (this includes G-Higgs bundles and $G^{\mathbb{R}}$-Higgs bundles). This is a correspondence between polystable pairs and solutions to a gauge-theoretical equation. We collect here the required ingredients and the statement of the correspondence in our case. This follows from the general Hitchin–Kobayashi correspondence for a (G,V)-Higgs pair [21, 22], and we refer to [9, Section 6.4] for an explanation adapted to our situation.

Let $(G_0, \mathfrak{g}_1)$ be a Vinberg θ-pair coming from the complex reductive group G with Lie algebra $\mathfrak{g}$. First we will explain the correspondence for G-Higgs bundles. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an antiholomorphic involution giving a compact real form $K \subseteq G$ (cf. Example 4) with Lie algebra $\mathfrak{k}$. Such involution always exists (see e.g. [36, Theorem 6.11]). Let E be a G-bundle and $h \in H^0(X, E(G/K))$ a reduction of the structure group to K. These reductions are also called **metrics** (since, if $G = \mathrm{GL}(n,\mathbb{C})$, then $K = U(n)$ and the reduction translates to a usual hermitian metric). As τ fixes K, we get a well defined involution $\tau'_h : E_h(\mathfrak{g}) \otimes K_X \to E_h(\mathfrak{g}) \otimes K_X$. Notice that since K_X is the holomorphic cotangent bundle of X, this is an involution of the space of holomorphic $E_h(\mathfrak{g})$-valued forms on X. Composing with conjugation of 1-forms gives a map $\tau_h : \Omega^{1,0}(X, E_h(\mathfrak{g})) \to \Omega^{0,1}(X, E_h(\mathfrak{g}))$. Given an element $\varphi \in \Omega^{1,0}(X, E_h(\mathfrak{g})) = H^0(X, E_h(\mathfrak{g}) \otimes K_X)$, we have $[\varphi, -\tau_h(\varphi)] \in \Omega^{1,1}(X, E_h(\mathfrak{k}))$, as it is fixed by τ_h.

On the other hand, a metric h induces a unique compatible connection, called *Chern connection*, whose curvature is $F_h \in \Omega^{1,1}(E_h(\mathfrak{k}))$ (see for example [26, Chapter III] for the vector bundle case). As X is a Riemann surface, it is Kähler and we can select a Kähler form $\omega \in \Omega^{1,1}(X)$.

Theorem 5 (Hitchin–Kobayashi correspondence for G-Higgs bundles)**.** *Let $\alpha \in i\mathfrak{z}^{\Gamma}$ be a stability parameter. A G-Higgs bundle (E, φ) is α-polystable if and only if there exists a metric h on E such that the equality*

$$F_h + [\varphi, -\tau_h(\varphi)] = -i\alpha\omega$$

of elements in $\Omega^{1,1}(X, E_h(\mathfrak{k}))$ holds.

Note that in the previous theorem we are using that α is in the centre of $\mathfrak{k}$ and thus it defines a global section of the associated bundle $E_h(\mathfrak{k})$. For the Vinberg θ-pair case, the statement is similar but the compact involution $\tau : \mathfrak{g} \to \mathfrak{g}$ needs to verify the extra condition $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$. This ensures that the maximal compact $K_0 \leq G_0$ is just $G_0 \cap K$ (with Lie algebra $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$, the centre of which we denote $\mathfrak{z}_0$), as well as that $[\varphi, -\tau(\varphi)] \in \Omega^{1,1}(X, E_h(\mathfrak{k}_0))$. Such a compact involution exists [51, Theorem 3.72]. We then have

Theorem 6 (Hitchin–Kobayashi correspondence for Higgs pairs associated to Vinberg θ-pairs)**.** *Let $\alpha \in i\mathfrak{z}_0^\Gamma$ be a stability parameter, $(G_0, \mathfrak{g}_1)$ a Vinberg θ-pair from $\theta \in \mathrm{Aut}_m(G)$ and $\tau : \mathfrak{g} \to \mathfrak{g}$ a compact involution with $\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}$. A $(G_0, \mathfrak{g}_1)$-Higgs pair (E, φ) is α-polystable if and only if there exists a metric h on E such that the equality*

$$F_h + [\varphi, -\tau_h(\varphi)] = -i\alpha\omega$$

of elements in $\Omega^{1,1}(X, E_h(\mathfrak{k}_0))$ holds.

We will also need the correspondence for Higgs pairs twisted by an arbitrary line bundle L over X (cf. Definition 12). The only difference is that we cannot interpret the involution τ_h' from before as a map between differential forms. However, if we fix an hermitian metric h_L in L we can identify L with L^*. In other words, we now have $\tau_h : E_h(\mathfrak{g}) \otimes L \to E_h(\mathfrak{g}) \otimes L^*$. For an element $\varphi \in H^0(X, E_h(\mathfrak{g}) \otimes L)$, we then have $[\varphi, -\tau_h(\varphi)] \in H^0(X, E_h(\mathfrak{k}_0))$. The statement becomes:

Theorem 7 (Hitchin–Kobayashi correspondence for L-twisted Higgs pairs associated to Vinberg θ-pairs)**.** *Let $\alpha \in i\mathfrak{z}_0^\Gamma$ be a stability parameter, L a line bundle over X with a choice of metric $h_L : L \xrightarrow{\sim} L^*$, $(G_0, \mathfrak{g}_1)$ a Vinberg θ-pair from $\theta \in \mathrm{Aut}_m(G)$ and $\tau : \mathfrak{g} \to \mathfrak{g}$ a compact involution with $\tau(\mathfrak{g}_i) = \mathfrak{g}_{-i}$. An L-twisted $(G_0, \mathfrak{g}_1)$-Higgs pair (E, φ) is α-polystable if and only if there exists a metric h on E such that the equality*

$$F_h + [\varphi, -\tau_h(\varphi)]\omega = -i\alpha\omega$$

of elements in $\Omega^{1,1}(X, E_h(\mathfrak{k}_0))$ holds.

3.2. The $\mathbb{C}^*$-action and Higgs bundles associated to Vinberg $\mathbb{C}^*$-pairs

Let G be a complex semisimple Lie group with $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. Recall from Section 2.1 that the connected subgroup $G_0 \le G$ corresponding to $\mathfrak{g}_0$ is reductive and acts on $\mathfrak{g}_1$ giving a prehomogeneous vector space $(G_0, \mathfrak{g}_1)$. In this section we will see how $(G_0, \mathfrak{g}_1)$-Higgs pairs arise naturally as fixed points of the action of the multiplicative group $\mathbb{C}^*$ defined on $\mathcal{M}(G)$ by scaling of the Higgs field:

$$\lambda \cdot (E, \varphi) = (E, \lambda\varphi).$$

This action is of great importance for understanding the moduli space $\mathcal{M}(G)$. For example, it provides a stratification of the moduli space in affine subvarieties, each given as the points whose limit when $\lambda \to 0$ is the same fixed point [30, Sections 2 and 3]. Moreover, under a suitable action with positive weights on the Hitchin base $\mathcal{A}$, the Hitchin system that will be explored in detail in Chapter 5 becomes $\mathbb{C}^*$-equivariant and hence fixed points lie inside the fibre $h^{-1}(0)$, which is known to be singular. Furthermore, by taking limit $\lambda \to 0$ the moduli space retracts to the fixed points and thus the topology of $\mathcal{M}(G)$ can be inferred from that of the fixed locus. All of these are some of the motivations to study the fixed points.

Notice that $G_0 \leq G$ and $\mathfrak{g}_1 \subseteq \mathfrak{g}$, so it makes sense to consider whether a G-Higgs bundle reduces to a Higgs pair for the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ (or, similarly, to $(G_0, \mathfrak{g}_k)$, but we consider only the first piece as explained in Remark 1). These give fixed points [5, Section 4.1]:

Proposition 8. *Let (E, φ) be a G-Higgs bundle that reduces to a $(G_0, \mathfrak{g}_1)$-Higgs pair. Then, $(E, \lambda\varphi) \simeq (E, \varphi)$ for all $\lambda \in \mathbb{C}^*$.*

Proof. We will write (E, φ) for the reduced $(G_0, \mathfrak{g}_1)$-Higgs pair and show that it is fixed by $\mathbb{C}^*$. The result for the original (E, φ) will then follow by extending the structure group (i.e. viewing E as a G-bundle and using the inclusion $E(\mathfrak{g}_1) \subseteq E(\mathfrak{g})$). Let $\zeta \in \mathfrak{g}_0$ be a grading element (cf. Proposition 2). Let $g := \exp(t\zeta) \in G_0$ for some $t \in \mathbb{C}$. The action of G_0 on the G_0-bundle E gives a holomorphic automorphism $f_g \in \mathrm{Aut}(E)$ by the rule $e \mapsto eg$. This automorphism turns $[(e, v)] \in E(\mathfrak{g}_1)$ into $[(eg, v)] = [(e, gv)]$. Now, $gv = \mathrm{Ad}(\exp(t\zeta))(v) = \exp(\mathrm{ad}(t\zeta))(v) = \exp(t)v$, where we used that $\mathrm{ad}(\zeta)(v) = v$ by definition of the grading element. Thus, we obtain via applying f_g that $(E, \varphi) \simeq (E, \exp(t)\varphi)$. Since $t \in \mathbb{C}$ was arbitrary, the proposition follows. □

In fact, Simpson proved that the converse is also true [49], meaning that every fixed point reduces to a $(G_0, \mathfrak{g}_k)$-Higgs pair. We will give an argument [30, Section 3.1] for classical groups using the vector bundle interpretation of G-Higgs bundles in that case, explained in Example 14. Suppose that $(E, \varphi) \simeq (E, \lambda\varphi)$ for all $\lambda \in \mathbb{C}^*$, where E is a rank n vector bundle and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$. This means that we have automorphisms $f_\lambda \in \mathrm{Aut}(E)$ satisfying the diagram

$$\begin{tikzcd} E \arrow[r, "f_\lambda"] \arrow[d, "\lambda\varphi"'] & E \arrow[d, "\varphi"] \\ E \otimes K_X \arrow[r, "f_\lambda"] & E \otimes K \end{tikzcd}$$

This is, in particular, a $\mathbb{C}^*$-action on each fibre of E, which allows to decompose $E|_x = E_0|_x \oplus \cdots \oplus E_{k-1}|_x$ in weight spaces (that is, there is some $w_j \in \mathbb{Z}$ such that $f_\lambda(v) = \lambda^{w_j} v$ for $v \in E_j|_x$). Globally, this gives a decomposition $E = E_0 \oplus \cdots \oplus E_{k-1}$ as a direct sum of subbundles. Now, given $v \in E_j$ we have that $f_\lambda(\varphi(v)) = \lambda^{-1}\varphi(f_\lambda(v)) = \lambda^{w_j - 1}\varphi(v)$. If we assume (by reordering the indices) that the largest weight is w_0 and $w_j = w_0 - j$, we get that $\varphi(E_j) \subseteq E_{j+1} \otimes K_X$. Thus, we have obtained that the pair (E, φ) is precisely a Higgs pair for the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ in Example 2. (This is clear, at least, when $G = \mathrm{SL}_n(\mathbb{C})$. For other classical groups the reasoning here is the same and Example 2 can be modified accordingly by adding extra structure to the vector spaces and considering quiver representations respecting said structure).

3.3. Cyclic Higgs bundles as Higgs bundles associated to Vinberg θ-pairs

Let G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$ and $\theta \in \mathrm{Aut}_m(G)$ an automorphism of order m. Recall from Section 2.2 that this gives a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_k$. In this section we will explain how Higgs pairs associated to the Vinberg θ-pair $(G_0, \mathfrak{g}_1)$ arise as fixed points of cyclic group actions in $\mathcal{M}(G)$. We follow [25, 20].

The cyclic group $\mu_m = \{z \in \mathbb{C}^* : z^m = 1\}$ acts in the moduli space $\mathcal{M}(G)$ as follows. On one hand, we can act on a principal G-bundle E through the order m automorphism of G, via the rule $E \mapsto \theta(E) := E \times_\theta G$. A Higgs field $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K_X)$ gets sent to $\theta(\varphi) \in H^0(X, \theta(E)(\mathfrak{g}) \otimes K_X)$. On the other hand, fixing a primitive m-th root of unity $\zeta \in \mu_m$, we can scale the Higgs field using ζ as in the previous section. The resulting action of μ_m is

$$\zeta^j \cdot (E, \varphi) = (\theta^j(E), \zeta^j\theta^j(\varphi)).$$

Recall that $G_0 \leq G$ is the connected subgroup corresponding to $\mathfrak{g}_0$. The fixed point subgroup G^θ contains G_0 as the connected component of the identity, and we can also consider the pair $(G^\theta, \mathfrak{g}_1)$, sometimes referred to as an *extended Vinberg θ-pair*, as it keeps the relevant properties of Vinberg θ-pairs discussed in Section 2.2. The map taking a $(G^\theta, \mathfrak{g}_1)$-Higgs pair (E, φ) to a G-Higgs bundle $(E \times_{G^\theta} G, \varphi)$ (using the fact that $\mathfrak{g}_1 \subseteq \mathfrak{g}$) defines a map of moduli spaces

$$\mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G),$$

whose image is contained in the fixed point locus $\mathcal{M}(G)^{\mu_m}$ by the action described above.

Remark 8. When G is simply connected, such as in the case $G = \mathrm{SL}_n(\mathbb{C})$ of Example 5, the fixed point locus G^θ is connected and thus $G_0 = G^\theta$. Otherwise, if we want to work with the Vinberg θ-pair $(G_0, \mathfrak{g}_1)$, we can use again the extension of structure group $(E, \varphi) \mapsto (E \times_{G_0} G^\theta, \varphi)$ to get a map $\mathcal{M}(G_0, \mathfrak{g}_1) \to \mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G)$ with image contained in the fixed point locus of the μ_m action.

Remark 9. The map $\mathcal{M}(G^\theta, \mathfrak{g}_1) \to \mathcal{M}(G)$ does not surject onto the fixed point locus in general, as there is a set of automorphisms which are equivalent, in a sense, to θ, so that each associated $\mathcal{M}(G^{\theta'}, \mathfrak{g}'_1)$ also maps to the fixed point locus. If one considers the images corresponding to each associated automorphism, they do cover the smooth locus of $\mathcal{M}(G)^{\mu_m}$. All these aspects about fixed point loci of finite order automorphisms in moduli spaces of G-Higgs bundles are studied in [25].

The elements in the image of $\mathcal{M}(G^\theta, \mathfrak{g}_1)$ in $\mathcal{M}(G)$ have been called **cyclic G-Higgs bundles** in the literature. The automorphism is sometimes made explicit by referring to these pairs as **θ-cyclic G-Higgs bundles**.

Example 20. In the particular case where θ is an *inner automorphism*, that is, one of the form $x \mapsto gxg^{-1}$ for some $g \in G$, we get that $(\theta(E), \theta(\varphi)) \simeq (E, \varphi)$, so that the action of μ_m is just $(E,\varphi) \mapsto (E,\zeta\varphi)$. This is the case of the automorphism considered in Example 5 given by cyclic quiver representations. A Higgs pair for the pair $(G_0, \mathfrak{g}_1)$ in this cyclic quiver case can be seen (in terms of vector bundles, using the fact that $G \subseteq \mathrm{GL}_n(\mathbb{C})$, as in Example 14) as a holomorphic vector bundle $E = E_0 \oplus \cdots \oplus E_{m-1}$ with trivial determinant, and a traceless section $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$ such that $\varphi(E_k) \subseteq E_{k+1} \otimes K_X$, with indices taken in $\mathbb{Z}/m\mathbb{Z}$. Note that the case of $SU(p,q)$-Higgs bundles, defined in a similar way to Example 16, corresponds to the situation of $m = 2$, that is, points with $(E,\varphi) \simeq (E,-\varphi)$.

CHAPTER 4

The Toledo invariant

4.1. Toledo invariant for cyclic Higgs bundles coming from a $\mathbb{Z}$-grading

Throughout this section we work in the setting of special cyclic gradings, which we now recall. Consider a semisimple complex Lie group G with Lie algebra $\mathfrak{g}$ which has a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ with grading element $\zeta \in \mathfrak{g}_0$. We take an integer $m \geq 2$ such that $\mathfrak{g}_i = 0$ for $|i| \geq m$, and consider the associated $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}/m\mathbb{Z}} \bar{\mathfrak{g}}_k$, where, for $k \in \{1, \dots, m-1\}$, we have $\bar{\mathfrak{g}}_k = \mathfrak{g}_k \oplus \mathfrak{g}_{k-m}$, and $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$. We also suppose that this associated grading comes from some $\theta \in \mathrm{Aut}_m(G)$ as explained in Section 2.2. In particular, the discussions will apply for Higgs pairs associated to cyclic quivers (Example 5), as well as for $G^{\mathbb{R}}$-Higgs bundles when $G^{\mathbb{R}}$ is of Hermitian type (Example 8).

Let B be an invariant bilinear form on $\mathfrak{g}$ (such as the Killing form) and let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then, $B|_{\mathfrak{t}\times\mathfrak{t}}$ is positive definite and we get a dual form B^* on $\mathfrak{t}^*$. Let $\gamma \in \mathfrak{t}$ be the longest root such that a conjugate of the root space $\mathfrak{g}_\gamma$ belongs to $\mathfrak{g}_1$ (each $\mathbb{Z}$-grading of a Lie algebra $\mathfrak{g}$ can be conjugated so that the graded pieces $\mathfrak{g}_i$ are direct sums of root spaces, as explained in Section 2.1.1).

Definition 16. The **Toledo character** $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by

$$\chi_T(x) = B(\zeta, x)B^*(\gamma, \gamma).$$

This is indeed a character, as $B(\zeta, [x, y]) = -B([x, \zeta], y) = B(0, y) = 0$. The constant factor $B^*(\gamma, \gamma)$ ensures that the definition does not depend on the choice of invariant bilinear form.

Now let E be a principal G_0-bundle. As in the definition of stability of Higgs pairs in section 3.1, we can define the degree of E associated to χ_T by selecting $q \in \mathbb{N}$ such that $q\chi_T$ lifts to a character $\tilde{\chi} : G_0 \to \mathbb{C}^*$, and setting

$$\deg_{\chi_T}(E) := \frac{1}{q} \deg(E \times_{\tilde{\chi}} \mathbb{C}^*).$$

If no such q exists, the degree can still be defined via differential geometric techniques as explained in section 3.1 and detailed in [5, Section 4.2].

Definition 17. Let (E,φ) be a $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair. We define the **Toledo invariant** of (E,φ) by

$$\tau(E,\varphi) := \deg_{\chi_T}(E).$$

Remark 10. The invariant has been defined by using the prehomogeneous vector space $(G_0,\mathfrak{g}_1)$. However, since the Higgs field takes values in $\mathfrak{g}_1 \oplus \mathfrak{g}_{1-m}$, it makes sense to consider what happens if we try to use instead the prehomogeneous vector space $(G_0,\mathfrak{g}_{1-m})$. Recall from Remark 1 that this space is of the form $(G_0,\mathfrak{g}'_1)$ for the graded subalgebra $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_{1-m} \oplus \mathfrak{g}_{m-1}$. The corresponding grading element here is $\zeta' = \frac{\zeta}{1-m}$. We also need to select a new longest root γ', now with the condition that (a conjugate of) the root space $\mathfrak{g}_{\gamma'}$ belongs in $\mathfrak{g}_{1-m}$. Thus, the new Toledo character is $\chi'_T(x) = B(\zeta',x)B^*(\gamma',\gamma') = \frac{1}{1-m}\frac{B^*(\gamma',\gamma')}{B^*(\gamma,\gamma)}\chi_T(x)$. Consequently, we obtain an alternative Toledo invariant

$$\tau'(E,\varphi) := \frac{1}{1-m}\frac{B^*(\gamma',\gamma')}{B^*(\gamma,\gamma)}\tau(E,\varphi).$$

Notice that the signs of τ and τ' differ.

Example 21. We will compute the value of the Toledo invariant for the $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair corresponding to the Vinberg θ-pair of Example 5. As explained in Example 20, these can be seen as vector bundles $E = E_0 \oplus \cdots \oplus E_{m-1}$ with a Higgs field $\varphi \in H^0(X,\mathrm{End}(E)\otimes K_X)$ such that $\varphi(E_k) \subseteq E_{k+1}\otimes K_X$, the indices taken in $\mathbb{Z}/m\mathbb{Z}$. We first compute the Toledo character. Recall that in this case the grading element for the prehomogeneous vector space $(G_0,\mathfrak{g}_1)$, using the notation of Example 2, is $\zeta \in \mathfrak{g}_0 = (\bigoplus_{j=0}^{m-1}\mathrm{End}(V_j))_0$ given by $\zeta|_{V_j} = (j-\alpha)\,\mathrm{Id}\,|_{V_j}$. In $\mathrm{SL}_n(\mathbb{C})$ we can choose the invariant form $B(X,Y) = \mathrm{tr}(XY)$. All the roots of $\mathrm{SL}_n(\mathbb{C})$ have the same length, which is 2 under this invariant form. Thus, we get, for $x = (f_0,\dots,f_{m-1}) \in \mathfrak{g}_0$, that

$$\chi_T(x) = B(\zeta,x)B^*(\gamma,\gamma) = 2\sum_{j=0}^{m-1}(j-\alpha)\,\mathrm{tr}(f_j).$$

A multiple $q\chi_T$ lifts to a character of the group G_0 if $2q(\alpha-j)$ is integral for all $j \in \{0,\dots,m-1\}$, resulting in

$$\tilde{\chi}(g) = \prod_{j=0}^{m-1}\det(g_j)^{2q(j-\alpha)},$$

for $g = (g_0,\dots,g_{m-1}) \in G_0$. Such q exists as α is rational. Then, we have the line bundle

$$E \times_{\tilde{\chi}} \mathbb{C}^* = \bigotimes_{j=0}^{m-1}\det(E)^{\otimes 2q(j-\alpha)},$$

so that, finally,

$$\tau(E,\varphi) = \frac{1}{q}\deg(E\times_{\tilde{\chi}}\mathbb{C}^*) = 2\sum_{j=0}^{m-1}(j-\alpha)\deg E_j.$$

The value of α, if explicitly computed, results in $\alpha = \frac{\sum_{j=0}^{m-1} jd_j}{\sum_{j=0}^{m-1} d_j}$, and we have that $d_j = \operatorname{rank} E_j$, so we see that the resulting invariant depends on the degrees and ranks of each piece E_j.

The case of $m = 2$ corresponds to $\mathrm{SU}(p,q)$-Higgs bundles defined as in Example 16. If we let (p,q) and (a,b) be the ranks and degrees, respectively, of each of the two pieces E_0 and E_1, the previous formula reads

$$\tau(E,\varphi) = 2\frac{pb - qa}{p+q},$$

which is the Toledo invariant for $\mathrm{SU}(p,q)$-Higgs bundles defined in [10]. Notice that working with $\mathrm{SU}(p,q)$ (instead of $\mathrm{U}(p,q)$) imposes the additional restriction $\deg(E_0) = -\deg(E_1)$, so we can substitute $b = -a$ in order to obtain a simplified expression for the invariant. Similarly, for larger values of m we can use the constraint $\sum_{j=0}^{m-1} \deg E_j = 0$ to obtain an expression with one parameter less.

4.2. The Arakelov–Milnor–Wood inequality

The goal of this section is to show that the Toledo invariant is bounded on the moduli space $\mathcal{M}^\alpha(G_0, \bar{\mathfrak{g}}_1)$, generalizing the existing inequalities for the Toledo invariant in the particular cases explained at the beginning of the chapter. We will use the fact that we are working with a Vinberg θ-pair which relates to the prehomogeneous vector spaces $(G_0, \mathfrak{g}_1)$ and $(G_0, \mathfrak{g}_{1-m})$ in order exploit some tools from the theory of these spaces. Before the main theorem, we will first need some definitions as well as further results about prehomogeneous vector spaces. We begin by associating a number to each element of $\mathfrak{g}_1$ depending on its orbit.

Definition 18. Let $e \in \mathfrak{g}_1$ and (h, e, f) a $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$ (cf. Proposition 3). We define the **Toledo rank of** e by

$$\operatorname{rank}_T(e) := \frac{1}{2}\chi_T(h).$$

This number is indeed independent of the representative of a given G_0-orbit: if $e, e' \in \mathfrak{g}_1$ belong to the same orbit and h, h' are the corresponding elements in $\mathfrak{g}_0$, by [5, Proposition 2.19] we get that there exists $g \in G_0$ such that $\operatorname{Ad}_g h = h'$. Then, by Ad-invariance of B as well as the fact that $\operatorname{Ad}_g \zeta = \zeta$, we have $\chi_T(h) = B(\zeta, h)B^*(\gamma,\gamma) = B(\operatorname{Ad}_g \zeta, \operatorname{Ad}_g h)B^*(\gamma,\gamma) = \chi_T(h')$. Moreover, by [5, Proposition 3.16], if $e' \in \Omega$ is an element of the open orbit, we have

$$0 \leq \operatorname{rank}_T(e) \leq \operatorname{rank}_T(e'),$$

with the second inequality becoming an equality if and only if $e \in \Omega$. In other words, the maximum value of the rank is given precisely by elements of the open orbit.

Definition 19. We define the **Toledo rank of** $(G_0, \mathfrak{g}_1)$ to be $\operatorname{rank}_T(G_0, \mathfrak{g}_1) := \operatorname{rank}_T(e)$ for any $e \in \Omega$.

Definition 20. Let $\rho : G \to \mathrm{GL}(V)$ be a prehomogeneous vector space and $\chi : G \to \mathbb{C}^*$ a character. A non-constant rational function $F : V \to \mathbb{C}$ is called a **relative invariant** for χ if, for all $g \in G$ and $v \in V$, we have

$$F(\rho(g) \cdot v) = \chi(g)F(v).$$

We have the following lemma [5, Proposition 2.8] that guarantees the existence of a relative invariant.

Lemma 1. *Suppose that $(G_0, \mathfrak{g}_1)$ is JM-regular. Then, there exists $q \in \mathbb{N}$ such that $q\chi_T$ lifts to a character $\tilde{\chi}_T : G_0 \to \mathbb{C}^*$ having a relative invariant F of degree $q \operatorname{rank}_T(G_0, \mathfrak{g}_1)$.*

Proof. Take $e \in \Omega \subseteq \mathfrak{g}_1$ and complete it to an $\mathfrak{sl}_2$-triple $(2\zeta, e, f)$ by JM-regularity. Consider its stabilizer $G_0^e \subseteq G_0$ with Lie algebra $\mathfrak{g}_0^e \subseteq \mathfrak{g}_0$. For $x \in \mathfrak{g}_0^e$ we have $\chi_T(x) = B(\zeta, x)B^*(\gamma, \gamma) = \frac{1}{2}B([e, f], x)B^*(\gamma, \gamma) = \frac{-1}{2}B(f, [e, x])B^*(\gamma, \gamma) = 0$. This means that some $q\chi_T$ lifts to the connected component of the identity of G_0^e as the trivial character, and since G_0^e has finitely many components, we can choose q so that $q\chi_T$ lifts to a character $\tilde{\chi}_T$ of G_0 with $\tilde{\chi}|_{G_0^e} = 1$. By [43, Proposition 19], this is a sufficient condition for the existence of a relative invariant $F : G_0 \to \mathbb{C}^*$, and $F(e) \neq 0$ since otherwise $F \equiv 0$ by density of Ω. Relative invariants are homogeneous functions by [43, Proposition 3], so we can obtain the degree by letting $t \in \mathbb{C}$ and computing

$$F(\exp(t)e) = F(\mathrm{Ad}(\exp(t\zeta))(e)) = \tilde{\chi}_T(\exp(t\zeta))F(e) = \exp(tqB(\zeta, \zeta)B^*(\gamma, \gamma))F(e),$$

so that the degree is $qB(\zeta, \zeta)B^*(\gamma, \gamma) = q \operatorname{rank}_T(G_0, \mathfrak{g}_1)$. □

The previous result needs a JM-regular space, but we will be able to apply it in general by starting from an arbitrary prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ and an element $e \in \mathfrak{g}_1$, and obtaining a JM-regular prehomogeneous vector subspace $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ such that $e \in \hat{\Omega} \subseteq \hat{\mathfrak{g}}_1$ where $\hat{\Omega}$ is the open orbit. The construction is as follows. Take an $\mathfrak{sl}_2$-triple (h, e, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$ using Proposition 3. By the theory of $\mathfrak{sl}_2$-representations, in particular [36, Corollary 1.72], $\mathrm{ad}(h)$ diagonalizes with integer eigenvalues, so we have another $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \tilde{\mathfrak{g}}_k$ given by the eigenspaces. Define

$$\hat{\mathfrak{g}}_k := \mathfrak{g}_k \cap \tilde{\mathfrak{g}}_{2k},$$

yielding the subalgebra $\hat{\mathfrak{g}} = \bigoplus_{k \in \mathbb{Z}} \hat{\mathfrak{g}}_k$.

Note that since $[h, \zeta] = 0$ we have $h, \zeta \in \hat{\mathfrak{g}}_0$, and since $\frac{1}{2}[h, e] = [\zeta, e] = 1$ we also have $e \in \hat{\mathfrak{g}}_1$. Also, $\hat{\mathfrak{g}}$ is precisely the subalgebra of elements where ζ and $\frac{h}{2}$ coincide, that is, the stabilizer of $s := \zeta - \frac{h}{2}$ in $\hat{\mathfrak{g}}$. If $\hat{G}_0 \subseteq G_0$ is the centralizer of h in G_0 (equivalently, the centralizer of s), then $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ is a prehomogeneous vector subspace of $(G_0, \mathfrak{g}_1)$ which is JM-regular and $e \in \hat{\Omega} \subseteq \hat{\mathfrak{g}}_1$, the latter by Malcev–Kostant theorem [36, Theorem 10.10].

Definition 21. Given $e \in \mathfrak{g}_1$, the prehomogeneous vector subspace $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ of $(G_0, \mathfrak{g}_1)$ constructed above is called a **maximal JM-regular prehomogeneous vector subspace** for e.

We will need the following lemma about $\mathfrak{sl}_2$-representations.

Lemma 2. *Let (h,e,f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $k \in \mathbb{N}$ such that $\mathrm{ad}(e)^k = 0$. Then $\mathrm{ad}(h)$ is diagonalizable and the eigenvalues of $\mathrm{ad}(h)$ are integers in $\{-k+1,\dots,k-1\}$.*

Proof. By [36, Theorem 1.67] we have that $\mathfrak{g}$ decomposes as a direct sum of irreducible $\mathfrak{sl}_2$-representations, so we can work with an irreducible representation $\mathfrak{g}' \subseteq \mathfrak{g}$. Then, by [36, Theorem 1.66] we have a basis $\{v_0,\dots,v_n\} \subseteq \mathfrak{g}'$ such that $\mathrm{ad}(e)(v_i) = i(n-i+1)v_{i-1}$. This means that $\mathrm{ad}(e)^n(v_n) = nv_0 \neq 0$, and $\mathrm{ad}(e)^{n+1} = 0$, so that $k \geq n+1$. From the same description we have $\mathrm{ad}(h)(v_i) = (n-2i)v_i$. Thus, the eigenvalues of $\mathrm{ad}(h)$ are in $\{-n,\dots,n\} \subseteq \{-k+1,\dots,k-1\}$. □

Recall that the Higgs field φ of a $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair is a holomorphic section of the vector bundle $E(\bar{\mathfrak{g}}_1) \otimes K_X = E(\mathfrak{g}_1 \oplus \mathfrak{g}_{1-m}) \otimes K_X$. By projecting to the $\mathfrak{g}_1$ and $\mathfrak{g}_{1-m}$ factors, respectively, we get $\varphi^+ \in H^0(X, E(\mathfrak{g}_1))$, $\varphi^- \in H^0(X, E(\mathfrak{g}_{1-m}))$ with $\varphi^+ + \varphi^- = \varphi$. Notice that (E,φ^+) and (E,φ^-) are Higgs pairs associated to the prehomogeneous vector spaces $(G_0,\mathfrak{g}_1)$ and $(G_0,\mathfrak{g}_{1-m})$, respectively. Define $\mathrm{rank}_T(\varphi^+) := \mathrm{rank}_T(\varphi^+(x))$ for a generic $x \in X$, and similarly for $\mathrm{rank}_T(\varphi^-)$ (this latter defined within the prehomogeneous vector space $(G_0,\mathfrak{g}_{1-m})$, meaning that in the computation the grading has to be changed first as in Remark 10). Now we are ready to state and prove the bounds for the Toledo invariant.

Theorem 8 (Arakelov–Milnor–Wood inequality)**.** *Let G be a complex semisimple Lie group, $m \in \mathbb{N}$ and $\theta \in \mathrm{Aut}_m(G)$. Suppose that the induced $\mathbb{Z}/m\mathbb{Z}$-grading, $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}/m\mathbb{Z}} \bar{\mathfrak{g}}_k$, coincides with the associated to a special $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k=-m+1}^{m-1} \mathfrak{g}_k$ with grading element $\zeta \in \mathfrak{g}_0$, that is, the relation $\bar{\mathfrak{g}}_k = \mathfrak{g}_k \oplus \mathfrak{g}_{k-m}$ holds for $k \in \{1,\dots,m-1\}$ and $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$. Let $G_0 \leq G$ be the connected subgroup corresponding to $\mathfrak{g}_0$ and $\alpha := \lambda\zeta$ for $\lambda \in \mathbb{R}$. Let γ be the longest root such that a conjugate of the root space $\mathfrak{g}_\gamma$ is contained in $\mathfrak{g}_1$.*

Then, if (E,φ) is an α-semistable $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair, the Toledo invariant $\tau(E,\varphi)$ satisfies the inequality

$$-\tau_L \leq \tau(E,\varphi),$$

where

$$\tau_L = \mathrm{rank}_T(\varphi^+)(2g-2) + \lambda(B^*(\gamma,\gamma)B(\zeta,\zeta) - \mathrm{rank}_T(\varphi^+)).$$

Moreover, if $m = 2$ or $\varphi^- = 0$, the upper bound

$$\tau(E,\varphi) \leq \tau_U,$$

where

$$\tau_U = \mathrm{rank}_T(\varphi^-)(2g-2) + \lambda(B^*(\gamma,\gamma)B(\zeta,\zeta) - \mathrm{rank}_T(\varphi^-)),$$

also holds.

Proof. Let $e \in \mathfrak{g}_1$ be an element in the orbit where φ^+ takes values generically and consider the associated $\mathfrak{sl}_2$-triple (h,e,f) with $h \in \mathfrak{g}_0$ given by Proposition 3. Let $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ be the maximal JM-regular prehomogeneous vector subspace for e from

Definition 21. In the following we will argue using the semistability of (E,φ), for the related definitions we refer to Section 3.1. Let $s := \zeta - \frac{h}{2}$ and consider the associated parabolic Lie algebra $\mathfrak{g}_{0,s} \subseteq \mathfrak{g}_0$ of the parabolic subgroup $P_{0,s} \leq G_0$ (the subscripts 0 are due to the group being G_0 and should not be confused with a superscript 0). We have a reduction $\sigma \in H^0(E(G_0/P_{0,s}))$: first, it is well defined over the $x \in X$ where $\varphi^+(x)$ is in the orbit of e (as $P_{0,s}$ is constructed from this value e), and since this happens generically, the section extends to X. Notice that the Toledo character can be split

$$\chi_T(x) = B^*(\gamma,\gamma)(B(\frac{h}{2},x) + B(s,x)),$$

so that, multiplying by an appropriate $q \in \mathbb{N}$, lifting to $\tilde{\chi}_T$ of the group, applying to the reduction E_σ, taking degree and dividing by q (the usual steps from previous definitions) we get:

$$\tau(E,\varphi) = \deg E(\sigma, B^*(\gamma,\gamma)s) + \deg_{\hat{\chi}_T}(E_\sigma), \tag{4.1}$$

where the rightmost term is the one corresponding to the character $\hat{\chi}_T : x \mapsto B^*(\gamma,\gamma)B(\frac{h}{2},x)$.

Now we will get bounds for both of these terms. The Levi factor $L_{0,s}$ of $P_{0,s}$ is the centralizer of s and thus it is $\hat{G}_0$. This means that the $P_{0,s}$-bundle E_σ can be further reduced by projecting to the Levi factor, obtaining $E_\sigma(\hat{G}_0)$ a $\hat{G}_0$-bundle with $\varphi^+ \in H^0(E_\sigma(\hat{G}_0)(\hat{\mathfrak{g}}_1) \otimes K_X)$ (and still $\varphi^- \in H^0(E_\sigma(\hat{G}_0)(\mathfrak{g}_{1-m}) \otimes K_X)$). Let $F : \hat{\mathfrak{g}}_1 \to \mathbb{C}$ be the relative invariant of Lemma 1 corresponding to the lift $\hat{\chi}_{T,q}$ of the character $q\hat{\chi}_T$. Notice that its degree is $q \operatorname{rank}_T(\hat{G}_0, \hat{\mathfrak{g}}_1) = q\hat{\chi}_T(\frac{h}{2})$. The relative invariant does not vanish at the open orbit $\hat{\Omega}$ (or else $F \equiv 0$), and φ^+ is generically in that orbit, so $F(\varphi^+)$ is a nonzero section of the line bundle $E_\sigma(\hat{\chi}_{T,q}) \otimes K_X^{q\hat{\chi}_T(\frac{h}{2})}$. This means that the degree of this line bundle is non-negative, namely

$$\deg_{\hat{\chi}_T}(E_\sigma) = \frac{1}{q}\deg(E_\sigma(\hat{\chi}_{T,q})) \geq -\hat{\chi}_T\left(\frac{h}{2}\right)(2g-2), \tag{4.2}$$

yielding a bound for one of the terms.

For the other term we use the α-semistability. On one hand, $[s,\hat{\mathfrak{g}}_1] = 0$ by definition, so $\varphi^+ \in H^0(E_\sigma(\hat{\mathfrak{g}}_{1,s}) \otimes K_X) \subseteq H^0(E_\sigma(\hat{\mathfrak{g}}_1) \otimes K_X)$. On the other hand, we also have $\mathfrak{g}_{1-m} \subseteq \mathfrak{g}_{1-m,s}$. In order to see this, take $x \in \mathfrak{g}_{1-m}$ and decompose it in $\mathrm{ad}(\frac{h}{2})$-eigenvectors $x = \sum_j x_j$ with $x_j \in \mathfrak{g}_{1-m}$ corresponding to the eigenvalue j. This is possible by Lemma 2, together with the fact that $\mathfrak{g}_{1-m}$ is $\mathrm{ad}(\frac{h}{2})$-invariant. From the same lemma we have $j \in \{1-m, \frac{1-2m}{2}, \dots, \frac{2m-1}{2}, m-1\}$. Thus,

$$[s,x_j] = [\zeta - \frac{h}{2}, x_j] = (1-m-j)x_j.$$

Since $1-m \leq j \leq m-1$, we have that $\lambda_j := (1-m-j) \leq 0$. Thus $\mathrm{Ad}(e^{ts})(x_j) = \exp(\mathrm{ad}(ts))(x_j) = e^{t\lambda_j}x_j$ is bounded as $t \to \infty$, and hence $x \in \mathfrak{g}_{1-m,s}$. In other words, we have seen that $\varphi^- \in H^0(E_\sigma(\mathfrak{g}_{1-m,s}) \otimes K_X)$ and thus $\varphi \in H^0(E_\sigma(\bar{\mathfrak{g}}_{1,s}) \otimes K_X)$. We can then apply the definition of α-semistability to get

$$\deg E(\sigma, B^*(\gamma,\gamma)s) \geq B(\lambda\zeta, B^*(\gamma,\gamma)s). \tag{4.3}$$

Combining equations (4.1), (4.2) and (4.3) results in

$$\tau(E,\varphi) \geq -\hat{\chi}_T\left(\frac{h}{2}\right)(2g-2) + \lambda B^*(\gamma,\gamma)B(\zeta,s).$$

What remains is just to rewrite $-\hat{\chi}_T(\frac{h}{2}) = B^*(\gamma,\gamma)B(\frac{h}{2},\frac{h}{2}) = B^*(\gamma,\gamma)(B(\zeta,\frac{h}{2}) + B(s,\frac{h}{2})) = B^*(\gamma,\gamma)(B(\zeta,\frac{h}{2}) + 0) = \operatorname{rank}_T(\varphi^+)$, as well as

$$\lambda B^*(\gamma,\gamma)B(\zeta,s) = \lambda(B^*(\gamma,\gamma)B(\zeta,\zeta) - B^*(\gamma,\gamma)B(\zeta,\frac{h}{2})) =$$

$$= \lambda(B^*(\gamma,\gamma)B(\zeta,\zeta) - \operatorname{rank}_T(\varphi^+)).$$

The upper bound is proven by replicating the argument with the prehomogeneous vector space $(G_0,\mathfrak{g}_{1-m})$ instead of $(G_0,\mathfrak{g}_1)$. This yields a lower bound for the Toledo invariant τ' of Remark 10, which is related to τ by a negative constant and hence provides an upper bound for τ. In order for the argument to work with $(G_0,\mathfrak{g}_{1-m})$, it is essential that the $\mathbb{Z}$-grading corresponding to this prehomogeneous vector space, which is the one given by $\frac{1}{1-m}\zeta$, also induces the Vinberg θ-pair $(G_0,\bar{\mathfrak{g}}_1)$. This happens when $m=2$, in which case we also have that $\tau' = -\tau$. If this is not the case, the only part of the argument that no longer works is the one using semistability, as it could be the case that $\varphi^+ \notin H^0(E_\sigma(\mathfrak{g}_{1,s})\otimes K_X)$. The $\varphi^- = 0$ case is established in [5, Theorem 5.3]. □

Remark 11. The previous bound applies for α-semistable pairs, so that in particular it holds in the moduli spaces $\mathcal{M}^\alpha(G_0,\bar{\mathfrak{g}}_1)$ of polystable elements. Moreover, the statement works for any stability parameter α which is a real multiple of the grading element. In the specially interesting particular case of $\alpha = 0$, we get that for any $(G_0,\bar{\mathfrak{g}}_1)$-semistable Higgs pair (E,φ) (in particular, for elements of $\mathcal{M}(G_0,\bar{\mathfrak{g}}_1)$) the following inequality is satisfied:

$$-(2g-2)\operatorname{rank}_T(\varphi^+) \leq \tau(E,\varphi).$$

By using the fact that $\operatorname{rank}_T(\varphi^+) \leq \operatorname{rank}_T(G_0,\mathfrak{g}_1)$, a (coarser) bound independent of the specific element is achieved:

$$-(2g-2)\operatorname{rank}_T(G_0,\mathfrak{g}_1) \leq \tau(E,\varphi).$$

Remark 12. In the case of $(G_0,\mathfrak{g}_1)$-Higgs pairs (recall from Section 3.2 that these arise as fixed points of a $\mathbb{C}^*$-action in the G-Higgs bundle moduli space) we can make the same definitions and, using the fact that an α-(semi,poly)stable $(G_0,\mathfrak{g}_1)$-Higgs pair is also α-(semi,poly)stable as a $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair, we obtain the same bound for the Toledo invariant. This is the result presented in [5].

Remark 13. For moduli spaces of $G^{\mathbb{R}}$-Higgs bundles, where $G^{\mathbb{R}} \subseteq G$ is a real form of Hermitian type (in other words, for the case $m=2$ in our framework), the corresponding $\mathbb{Z}$-grading still induces the Vinberg θ-pair $(G_0,\bar{\mathfrak{g}}_1)$ and we obtain the upper bound of Theorem 8. For $\alpha = 0$ this results in an upper bound

$$\tau(E,\varphi) \leq (2g-2)\operatorname{rank}_T(\varphi^-),$$

as well as combined coarser bound independent of the specific element

$$|\tau(E,\varphi)| \leq (2g-2)\operatorname{rank}(G_0,\mathfrak{g}_1).$$

This is the result presented in [6] generalizing previous specializations such as [10] for $G^{\mathbb{R}} = \mathrm{SU}(p,q)$.

Example 22. As a first example we derive the bound for $\mathrm{SU}(p,q)$-Higgs bundles (recall that this is the case of Higgs pairs associated to the Vinberg θ-pair in Example 5 when $m = 2$). We will assume with no loss of generality that $p \leq q$. The corresponding prehomogeneous vector space $(G_0,\mathfrak{g}_1)$ is the one in Example 2 for $m = 2$. A $\mathrm{SU}(p,q)$-Higgs bundle can then be seen in terms of vector bundles (cf. Example 20) as a pair (E,φ) where $E = E_0 \oplus E_1$ is a direct sum of holomorphic vector bundles over X with $\det E \simeq \mathcal{O}$, $\operatorname{rank} E_0 = p$, $\operatorname{rank} E_1 = q$ (we will also denote $\deg E_0 = a$, $\deg E_1 = b = -a$), and $\varphi : E \to E \otimes K_X$ is a traceless holomorphic vector bundle homomorphism with $\varphi(E_0) \subseteq E_1 \otimes K_X$ and $\varphi(E_1) \subseteq E_0 \otimes K_X$. The corresponding $\varphi^+ : E_0 \to E_1 \otimes K_X$ is the restriction.

In order to compute the Toledo rank, we can first look at an element $e \in \mathfrak{g}_1$. Recall from Example 2 that this is a linear map from V_0 to V_1 and its orbit consists of all such maps with coinciding rank, which we denote as $r \in \{0,\dots,p\}$. For example, we can take the representative given by $\begin{pmatrix} \mathrm{Id}_r \\ 0 \end{pmatrix}$ in some fixed basis. The corresponding $h \in \mathfrak{g}_0 = (\operatorname{End}(V_0) \oplus \operatorname{End}(V_1))_0$ from Jacobson–Morozov has matrix $\begin{pmatrix} -\mathrm{Id}_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ with respect to the same basis. Using the expression for the Toledo character of Example 21, this means that $\chi_T(h) = -2\alpha \cdot (-r) + 2(1-\alpha)r = 2r$, so that

$$\operatorname{rank}_T(e) = \frac{1}{2}\chi_T(h) = r,$$

showing that in this example the Toledo rank matches the usual rank of a linear map. Thus, the Arakelov–Milnor–Wood inequality reads

$$-\operatorname{rank}(\varphi^+)(2g-2) \leq 2\frac{pb-qa}{p+q} \leq \operatorname{rank}(\varphi^-)(2g-2),$$

where rank means the usual rank of a vector bundle homomorphism. The coarse version is $2\left|\frac{pb-qa}{p+q}\right| \leq p(2g-2)$. Both of these were obtained in [10, Lemma 3.24, Remark 3.29] using vector bundle techniques, which we will now explain as a preamble for the next example.

First, we have from [10, Remark 3.8] that the stability conditions for $\mathrm{SU}(p,q)$-Higgs bundles in terms of vector bundles are the same as those for classical complex groups explained in Example 18. Suppose that (E,φ) is semistable. If $\varphi^+ = 0$ then E_0 is φ-invariant and thus $\mu(E_0) \leq \mu(E)$. Otherwise, let $N := \ker(\varphi^+) \subseteq E_0$ and $I := \operatorname{Im}(\varphi^+) \otimes K_X^{-1} \subseteq E_1$. The first isomorphism theorem gives that $\operatorname{rank}(N) + \operatorname{rank}(I) = p$

as well as that φ^+ defines a nonzero section of $\det((E_0/N)^* \otimes I \otimes K_X)$, so the degree of that line bundle is non-negative, namely

$$\deg(N) + \deg(I) + \operatorname{rank}(I)(2g-2) \geq \deg(E_0).$$

Now, N and $E_0 \oplus I$ are both φ-invariant, so $\mu(N) \leq \mu(E)$ and $\mu(E_0 \oplus I) \leq \mu(E)$. Combining all these and rearranging results in

$$p(\mu(E_0) - \mu(E)) \leq \operatorname{rank}(I)(g-1) = \operatorname{rank}(\varphi^+)(g-1).$$

As remarked at the beginning of the argument, if $\varphi^+ = 0$ the above inequality still holds, and it is precisely the Arakelov–Milnor–Wood inequality.

Example 23. In the case of Higgs pairs associated to the Vinberg θ-pair given in Example 5 of cyclic quiver representations, it is possible to argue in a similar way to previous example, albeit the computations are more cumbersome. Suppose that (E,φ) is such a Higgs pair, seen in terms of vector bundles as described in Example 20. The orbit of an element $e = (e_0, \dots, e_{m-2}) \in \mathfrak{g}_1$ is given by all other elements $(e'_0, \dots, e'_{m-2})$ such that $\operatorname{rank} e_i \circ \dots e_{i+k} = \operatorname{rank} e'_i \circ \dots \circ e'_{i+k}$ for all $0 \leq i < i+k \leq m-2$. Denote by $n_{ij} := \operatorname{rank}(e_{j-1} \circ \dots \circ e_i)$ for $0 \leq i < j \leq m-1$. Any of those elements can be constructed by fixing a basis on each V_i and partitioning the resulting basis of V into Jordan blocks in the way that produces those ranks. From [1, Section 2.3], the number of Jordan blocks that start at V_i and end at V_j is given by

$$\lambda_{ij} := n_{ij} - n_{i-1,j} - n_{i,j+1} + n_{i-1,j+1},$$

where $n_{ij} = 0$ if $i < 0$ or $j \geq m$. From the Jordan blocks it is possible to obtain the associated h as explained at the end of Example 2. This shows that on a Jordan block starting at V_i and ending at V_j, thus of length $j-i+1$, we have $h(u_l) = -(j-i-2l)u_l$ for $l \in \{0, \dots, j-i\}$, where $u_l \in V_{i+l}$. This means that

$$\operatorname{rank}_T(e) = B(\zeta, h) = \sum_{0 \leq i < j \leq m-1} \lambda_{ij} \sum_{l=0}^{j-i} (\alpha - i - l)(j - i - 2l).$$

This depends on the ranks of the consecutive compositions of the maps from E_j to E_{j+1} given by φ^+.

As an example, consider the case where $\operatorname{rank} E_j = k$ for all j, where $mk = n$. We will compute the coarse bound, that is, the value of $\operatorname{rank}_T(G_0, \mathfrak{g}_1)$, which equals the value of the Toledo character at the element $e = (e_0, \dots, e_{m-2}) \in \mathfrak{g}_1$ given in some basis by $e_j = \operatorname{Id}_k$. The corresponding $h = (h_0, \dots, h_{m-1})$, with each $h_j \in \operatorname{End}(V_j)$ given in the same basis by $-(m-1-2j)\operatorname{Id}_k$, and the Toledo rank turns out to be $\operatorname{rank}_T(G_0, \mathfrak{g}_1) = \frac{km(m-1)(m+1)}{6}$. The rank can also be computed via the previous general method, noticing that in this case the only nonzero λ_{ij} is $\lambda_{0,m-1} = k$. The result is the same. The coarse inequality is then

$$-(2g-2)\frac{km(m+1)(m-1)}{6} \leq \tau(E, \varphi).$$

This inequality can be deduced using the same vector bundle techniques from previous example. We will assume that each $\varphi_j^+ : E_j \to E_{j+1} \otimes K_X$ is nonzero for simplicity, the zero case being treated using that E_j is invariant as before. We denote $N_j := \ker(\varphi_j^+) \subseteq E_j$, $I_j := \mathrm{Im}(\varphi_j^+)K_X^{-1} \subseteq E_{j+1}$. We have $\operatorname{rank} N_j + \operatorname{rank} I_j = k$, as well as a nonzero section of $\det((E_j/N_j)^* \otimes I_j \otimes K_X)$ yielding

$$\deg N_j + \deg I_j + \operatorname{rank} I_j(2g-2) \geq \deg E_j.$$

We apply the semistability condition to the invariant subbundles N_j and $I_j \oplus \bigoplus_{i \neq j+1} E_i$, resulting in the inequalities

$$\deg N_j \leq \mu(E) \operatorname{rank}(N_j) = \mu(E)(k - \operatorname{rank} I_j),$$

$$\deg I_j + \sum_{i \neq j+1} \deg E_i \leq \mu(E)(\operatorname{rank} I_j + (m-1)k).$$

Summing these last two results in

$$\deg N_j + \deg I_j + \sum_{i \neq j+1} \deg E_i \leq \mu(E)mk.$$

Using the bound on $\deg N_j + \deg I_j$ we get

$$\deg E_j + \sum_{i \neq j+1} \deg E_i \leq \mu(E)mk + \operatorname{rank} I_j(2g-2) \leq \mu(E)mk + k(2g-2).$$

Dividing both sides by k and then subtracting $\mu(E)m$ we get the following in terms of slopes:

$$(\mu(E_j) - \mu(E)) + \sum_{i \neq j+1} (\mu(E_i) - \mu(E)) \leq 2g-2.$$

We will rewrite this for convenience as:

$$2(\mu(E_j) - \mu(E)) + \sum_{i \notin \{j, j+1\}} (\mu(E_i) - \mu(E)) \leq 2g-2.$$

Using the identity $\mu(E_j) - \mu(E) = \sum_{i \neq j}(\mu(E) - \mu(E_i))$, which is easy to check, we also obtain

$$2(\mu(E) - \mu(E_{j+1})) + \sum_{i \notin \{j, j+1\}} (\mu(E) - \mu(E_i)) \leq 2g-2.$$

Summing the last two inequalities gives

$$\mu(E_j) - \mu(E_{j+1}) \leq 2g-2.$$

Fix $l \in \{1, \ldots, m-1\}$. We recall that we are working with subindices mod m. Chaining the previous inequality several times results in

$$\mu(E_{j-l}) - \mu(E_j) \leq l(2g-2).$$

Note that the expression of the Toledo invariant from Example 21 can be written in terms of the slopes (as $\deg E_j = k\mu(E_j)$). It is possible to see by comparing the coefficients of each slope that the expression can be grouped as

$$\tau(E,\varphi) = -\frac{2k}{m}\sum_{l=1}^{m-1}\sum_{j=0}^{m-1-l} l(\mu(E_{j-l}) - \mu(E_j)).$$

The previous bound on differences of slopes gives the desired

$$\tau(E,\varphi) \geq -\frac{2k}{m}\sum_{l=1}^{m-1}\sum_{j=0}^{m-1-l} l^2(2g-2) = -(2g-2)\frac{km(m-1)(m+1)}{6}.$$

4.3. Maximal cyclic Higgs bundles

In this section we study the locus $\mathcal{M}^{\max}(G_0,\bar{\mathfrak{g}}_1) \subseteq \mathcal{M}(G_0,\bar{\mathfrak{g}}_1)$ of *maximal Higgs pairs*, meaning Higgs pairs (E,φ) such that $\tau(E,\varphi) = -(2g-2)\,\mathrm{rank}_T(G_0,\mathfrak{g}_1)$, the lowest possible according to Theorem 8. We do so in the case where $(G_0,\mathfrak{g}_1)$ is JM-regular. This is the generalization of the *tube type* situation for Hermitian real forms studied in [6]. We have the following observation:

Proposition 9. *Suppose that $(G_0,\mathfrak{g}_1)$ is JM-regular. A polystable $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair (E,φ) is maximal if and only if $\varphi^+(x) \in \Omega \subseteq \mathfrak{g}_1$ for all $x \in X$, where Ω is the open orbit.*

Proof. First, if (E,φ) is maximal we need that $\mathrm{rank}(\varphi^+) = \mathrm{rank}_T(G_0,\mathfrak{g}_1)$ as we have $\tau \geq -\mathrm{rank}_T(\varphi^+)(2g-2)$. This implies that φ^+ is in Ω generically. With the notation from the proof of Theorem 8, by JM-regularity we have $s=0$ and the only thing that we have to inspect is when does $\deg_{\hat{\chi}_T}(E_\sigma) = -\hat{\chi}_T(\frac{h}{2})(2g-2)$ hold. This happens if and only if the degree of the line bundle $E_\sigma(\hat{\chi}_{T,q}) \otimes K_X^{q\hat{\chi}_T(\frac{h}{2})}$ is zero, which happens if and only if its nonzero section $F(\varphi^+)$ is nonvanishing, meaning that $\varphi^+(x) \in \Omega$ for all $x \in X$. □

The space of maximal cyclic bundles is nonempty because already in the case of fixed points of the $\mathbb{C}^*$-action [5] there are polystable $(G_0,\mathfrak{g}_1)$-Higgs pairs attaining this bound. We recall the construction.

Let $e \in \Omega \subseteq \mathfrak{g}_1$ be an element of the open orbit and complete it to an $\mathfrak{sl}_2$-triple (h,e,f) with $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-1}$ using Proposition 3. Let $S \leq G_0$ be the connected subgroup with Lie algebra $\langle e,f,g\rangle \subseteq \mathfrak{g}$, which can be isomorphic to $\mathrm{PSL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$ depending on G and the triple. Let $C \leq G$ be the reductive group centralizing $\{h,e,f\}$, which coincides with $G_0^e \leq G_0$: it has to be contained in G_0 in order to stabilize h and it should also stabilize e, and this is sufficient. Let $T \leq S$ be the connected subgroup with Lie algebra $\langle h\rangle$. We have $T \leq G_0$ as $h \in \mathfrak{g}_0$. There are two cases:

- If $S \simeq \mathrm{SL}_2(\mathbb{C})$, it is simply connected and thus the representation $\langle h\rangle \to \mathfrak{gl}(\langle e\rangle)$ given by $\lambda h \mapsto \mathrm{ad}(\lambda h)$ lifts to a representation $\mathbb{C}^* \simeq T \to \mathrm{GL}(\langle e\rangle)$, this being just the adjoint representation. As $[h,e]=2e$, the $\mathbb{C}^*$-action we get on $\langle e\rangle$ via this lift is $\lambda\cdot e=\lambda^2 e$. Choose a square root $K_X^{\frac{1}{2}}$ (this can be done as $\deg K_X = 2g-2$ is even) and let E_T be the frame bundle for $K_X^{-\frac{1}{2}}$, in other words, the $\mathbb{C}^*$-bundle such that the bundle associated to the standard representation of $\mathbb{C}^*$ in $\mathbb{C}$ is $E_T(\mathbb{C})\simeq K_X^{-\frac{1}{2}}$. Using the isomorphism $\mathbb{C}^*\simeq T$ we have a T-bundle, and since T acts on $\langle e\rangle$ with weight 2 we have $E_T(\langle e\rangle)\simeq (K_X^{-\frac{1}{2}})^2=K_X^{-1}$. This means that $E_T(\langle e\rangle)\otimes K_X\simeq\mathcal{O}$ so we can define a constant section $e\in H^0(X,E_T(\langle e\rangle)\otimes K_X)$. Extending the structure group gives $(E_T(G_0),e)$ a $(G_0,\langle e\rangle)$-Higgs pair which is in particular a $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair.

- If $S\simeq \mathrm{PSL}_2(\mathbb{C})$, we can take its universal cover $\mathrm{SL}_2(\mathbb{C})\to S$ which is of degree two. The torus $T\subseteq S$ lifts to $\hat{T}\subseteq \mathrm{SL}_2(\mathbb{C})$. We have that $\hat{T}$ is a double cover of T and there are isomorphisms with $\mathbb{C}^*$ such that map $\hat{T}\simeq\mathbb{C}^*\to T\simeq\mathbb{C}^*$ is given by $\lambda\mapsto\lambda^2$. By the previous argument the adjoint action of $\hat{T}\simeq\mathbb{C}^*$ on $\langle e\rangle$ is given by $\lambda\cdot e=\lambda^2 e$, so that it descends to T as $\lambda\cdot e=\lambda e$. Now we let E_T be the frame bundle of K_X^{-1}. The associated bundle is $E_T(\langle e\rangle)\simeq K_X^{-1}$ and hence $E_T(\langle e\rangle)\otimes K_X\simeq\mathcal{O}$. Thus e defines a holomorphic section of $E_T(\langle e\rangle)\otimes K_X$. Extending the structure group gives $(E_T(G_0),e)$ a $(G_0,\langle e\rangle)$-Higgs pair and in particular a $(G_0,\bar{\mathfrak{g}}_1)$-Higgs pair.

The resulting $(E_T(G_0),e)$ is called **uniformising Higgs bundle** and is polystable (see [34]). By construction $\varphi^+=e$ is always in the open orbit, hence its Toledo invariant attains the bound.

Example 24. Consider the case of Higgs pairs associated to the Vinberg θ-pair in Example 5, whose prehomogeneous vector space $(G_0,\mathfrak{g}_1)$ was described in Example 2. We will assume that $\operatorname{rank} V_j=k$ for all j, which is a situation that we know is JM-regular. Take $e=(e_0,\dots,e_{m-2})\in\mathfrak{g}_1$ to be the one with matrix $e_j=\mathrm{Id}_k$ in some basis of V. We know that in this case $h|_{V_j}=-(m-1-2j)\,\mathrm{Id}_k$, and hence the torus T is given by the elements $t_\lambda\in G_0$ for $\lambda\in\mathbb{C}^*$ defined by $t_\lambda|_{V_j}=\lambda^{-(m-1-2j)}\,\mathrm{Id}_k$. It is easy to check that $t_\lambda e t_{\lambda^{-1}}=\lambda^2 e$ so we are in the situation of $S\simeq\mathrm{SL}_2(\mathbb{C})$, and hence E_T is the frame bundle of $K_X^{-\frac{1}{2}}$. The extension $E_T(G_0)$ is obtained by using the weights $-(m-1-2j)$ on each V_j explained before, resulting in the frame bundle for the vector bundle (which we denote in the same way) $E_T(G_0)=\bigoplus_{j=0}^{m-1}(K_X^{\frac{m-1}{2}-j})^{\oplus k}$. The Higgs field is given by e and thus it is the holomorphic map which on the i-th summand of the form $K_X^{\frac{m-1}{2}-j}$ maps to the i-th summand of the form $K_X^{\frac{m-1}{2}-j-1}$ as *the identity*, using the fact that $(K_X^{\frac{m-1}{2}-j})^*\otimes K_X^{\frac{m-1}{2}-j-1}\otimes K_X\simeq\mathcal{O}$ so we can take a well-defined global identity map.

Notice that this is a direct sum of k copies of the uniformising bundle for $k=1$, which is precisely the point above 0 in the Hitchin section that will be described at the end of Section 5.1.

Our goal for the remainder of the section is to analyse in our context the *Cayley correspondence* that takes place in the Hermitian case [6, Section 5], in the prehomogeneous vector space case [5, Section 6] and in the case of real forms induced by *magical* $\mathfrak{sl}_2$*-triples*, which are a generalization of the real forms of Hermitian type [9]. This is a correspondence between the locus of polystable Higgs pairs with maximal Toledo invariant, and the moduli space of K_X^m-twisted polystable (C, V)-Higgs pairs, where $C = G_0^e$ is the same as above and $V \subseteq \mathfrak{g}_0$ is certain vector subspace, the action of C in V being the adjoint. First we will identify this subspace in our situation.

Consider the $\mathfrak{sl}_2\mathbb{C}$-representation on $\mathfrak{g}$ given by the triple (h, e, f). By [36, Theorem 1.67] we have that $\mathfrak{g}$ decomposes as a direct sum of irreducible $\mathfrak{sl}_2\mathbb{C}$-representations, each of them uniquely determined by its dimension [36, Theorem 1.66]. Let $W \subseteq \mathfrak{g}$ be the direct sum of all the subspaces corresponding to irreducible representations of dimension $2m-1$. Notice that this is the maximum possible such dimension, since $e \in \mathfrak{g}_1$ implies $\mathrm{ad}(e)^{2m} \equiv 0$. The desired subspace is $V := \mathfrak{g}_0 \cap W \subseteq \mathfrak{g}_0$.

Proposition 10. *Suppose that the prehomogeneous vector space* $(G_0, \mathfrak{g}_1)$ *is JM-regular. The map* $\mathrm{ad}(e)^{m-1} : \mathfrak{g}_{1-m} \to V$ *is an isomorphism.*

Proof. By the structure of irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ ([36, Theorem 1.66]) and the fact that the maximum possible dimension of an irreducible representation is $2m-1$, the eigenvalue $2(1-m)$ for $\mathrm{ad}(h)$ can only appear on elements of W and hence we have that $\ker(\mathrm{ad}(h) - 2(1-m)\,\mathrm{Id}) \subseteq W$. By JM-regularity, $h = 2\zeta$ and thus this space is precisely $\mathfrak{g}_{1-m}$, i.e. $\mathfrak{g}_{1-m} \subseteq W$. As W is a subrepresentation, we have for all j that $\mathrm{ad}(e)^j(W) \subseteq W$. Moreover, since $e \in \mathfrak{g}_1$, we have $\mathrm{ad}(e)^{m-1}(\mathfrak{g}_{1-m}) \subseteq \mathfrak{g}_0$. We conclude that $\mathrm{ad}(e)^{m-1}(\mathfrak{g}_{1-m}) \subseteq V$ so the map is well defined.

It is an isomorphism: suppose that in W there are n summands of the irreducible $\mathfrak{sl}_2\mathbb{C}$-representation of dimension $2m-1$. Let $B := \{v_1, \ldots, v_n\}$ be n linearly independent eigenvectors for the eigenvalue $2(1-m)$ of $\mathrm{ad}(h)$, one in each irreducible representation. We know from the previous argument that $\mathfrak{g}_{1-m} = \langle B \rangle$. Now, $B' := \{\mathrm{ad}(e)^{m-1}(v_1), \ldots, \mathrm{ad}(e)^{m-1}(v_n)\}$ are linearly independent (because each of them is in a different irreducible representation) and by the structure of irreducible representations we have $\langle B' \rangle = V$. □

Moreover, it makes sense to consider the pair (C, V) as the action of C leaves V invariant. This is because of the fact that if $\mathfrak{c} \subseteq \mathfrak{g}_0$ is the Lie algebra of C, and we take $c \in \mathfrak{c}$, $\mathrm{ad}^{m-1}(e)(x) \in V$ (where $x \in \mathfrak{g}_{1-m}$), we have that $[c, \mathrm{ad}^{m-1}(e)(x)] = \mathrm{ad}^{m-1}(e)([c, x]) \in V$, where we used that $[c, e] = 0$ (recall that C centralizes e) and that $[c, x] \in \mathfrak{g}_{1-m}$.

Finally, suppose that E_C is a principal C-bundle and recall the T-bundle E_T from before. Since $m : T \times C \to G_0$ is a group homomorphism (because T commutes with C, this follows from the fact that if $x \in \mathfrak{g}_0$ centralizes e, then $[e, [h, x]] = [[e, h], x] + [h, [e, x]] = [-2e, x] + 0 = 0$), it is possible to define the G_0-bundle

$$(E_T \otimes E_C)(G_0) := (E_T \times E_C) \times_m G_0.$$

This is a notion of *tensor product* for principal bundles, and it works similarly for any two commuting subgroups of G_0 (for example, we can tensor any G_0-bundle by any bundle for a central subgroup such as T). As with vector bundles, given metrics h_T and h_C on each bundle respectively, there is a well defined product metric $h_T \otimes h_C$ on $E_T \otimes E_C(G_0)$ and the curvatures of the Chern connections verify $F_{h_T \otimes h_C} = F_{h_T} + F_{h_C}$.

We can now state the Cayley correspondence in this case. We need an extra assumption on (C, V) in order to prove the full correspondence, which is that it be a Vinberg θ-pair. This condition is not automatic.

Theorem 9 (Cayley correspondence)**.** *Suppose that the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is JM-regular. There is an injective map*

$$\mathcal{M}^{\max}(G_0, \bar{\mathfrak{g}}_1) \to \mathcal{M}_{K_X^m}(C, V).$$

Moreover, if the pair (C, V) is a Vinberg θ-pair, the map is surjective.

Proof. First we will define the map, which is a version of the *global Slodowy slice map* from [12] which, in turn, generalises the map behind the Cayley correspondence for hermitian forms [6] and magical $\mathfrak{sl}_2$-triples [9]. The definition uses the notation and elements introduced throughout the section, and is an extension of the construction given in [5] for the case $\varphi^- = 0$. Take $(E, \varphi) \in \mathcal{M}^{\max}(G_0, \bar{\mathfrak{g}}_1)$. We will use the fact that $E(\mathfrak{g}_1) \otimes K_X \simeq (E_T^{-1} \otimes E)(\mathfrak{g}_1)$, which is true in both cases $S \simeq \mathrm{SL}_2(\mathbb{C})$ or $S \simeq \mathrm{PSL}_2(\mathbb{C})$ as can be seen, for example, checking that the transition functions for both agree.

Since the Toledo invariant is maximal, by Proposition 9 we have $\varphi^+(x) \in \Omega$ for all $x \in X$. Thus, $\varphi^+ \in H^0(X, E(\Omega) \otimes K_X) = H^0(X, (E_T^{-1} \otimes E)(\Omega))$. As C is the stabilizer of the element $e \in \Omega$ in G_0, we have $\Omega = G_0/C$. In other words, $\varphi^+ \in H^0((E_T^{-1} \otimes E)(G_0/C))$ is a reduction of the structure group of $E_T^{-1} \otimes E$ from G_0 to C. Let E_C be the reduced C-bundle. This is the first part of the desired (C, V)-Higgs pair. Notice that we have $E_C(G_0) \simeq E_T^{-1} \otimes E$ which in turn gives the relation

$$E \simeq E_T \otimes E_C(G_0) = (E_T \otimes E_C)(G_0).$$

For the second part, we have as before that $(E_T \otimes E_C)(\langle e \rangle) \otimes K_X = E_C(\langle e \rangle) \otimes K_X^{-1} \otimes K_X = E_C(\langle e \rangle)$. Since C centralises e, there is a well-defined constant section $e \in H^0(X, E_C(\langle e \rangle)) = H^0(X, (E_T \otimes E_C)(\langle e \rangle) \otimes K_X) \subseteq H^0(X, (E_T \otimes E_C)(\mathfrak{g}_1) \otimes K_X)$. Using this section together with Proposition 10 gives a vector bundle isomorphism:

$$\mathrm{ad}(e)^{m-1} : (E_T \otimes E_C)(\mathfrak{g}_{1-m}) \otimes K_X \to (E_T \otimes E_C)(V) \otimes K_X^m.$$

Since T acts trivially on $\mathfrak{g}_0$ (because $[h, \mathfrak{g}_0] = 0$ by JM-regularity) we get that $(E_T \otimes E_C)(V) \simeq E_C(V)$. We can now take $\varphi^- \in H^0(X, E(\mathfrak{g}_{1-m}) \otimes K_X) = H^0(X, (E_T \otimes E_C)(G_0)(\mathfrak{g}_{1-m}) \otimes K_X) = H^0(X, (E_T \otimes E_C)(\mathfrak{g}_{1-m}) \otimes K_X)$ and apply the previous to get

$$\varphi' := \mathrm{ad}(e)^{m-1}(\varphi^-) \in H^0(X, E_C(V) \otimes K_X^m).$$

The Cayley correspondence map is then

$$(E, \varphi) \mapsto (E_C, \varphi').$$

The inverse has already been hinted at throughout the proof, but we collect it now. Given (E_C, φ') a (C, V)-Higgs pair, we set

$$E := (E_T \otimes E_C)(G_0),$$

$$\varphi^+ := e \in H^0(X, E(\mathfrak{g}_1) \otimes K_X),$$
$$\varphi^- := (\mathrm{ad}(e)^{m-1})^{-1}(\varphi') \in H^0(X, E(\mathfrak{g}_{1-m}) \otimes K_X).$$

Its Toledo invariant is the desired one (we know this if we start with (E_C, φ') which is the image of some (E, φ) with maximal invariant, but we shall see that it holds no matter the starting (C, V)-pair (E_C, φ')). We cannot simply argue that $\varphi^+ \in \Omega$ and use Proposition 9 because we do not yet know whether $(E, \varphi^+ + \varphi^-)$ is polystable. However, we can compute the invariant. First notice that if $\mathfrak{c} \subseteq \mathfrak{g}_0$ is the Lie algebra for C, the Toledo character vanishes: $\chi_T(\mathfrak{c}) \equiv 0$. This is because if $c \in \mathfrak{c}$, we have $2B(\zeta, c) = B(h, c) = B([e, f], c) = -B(f, [e, c]) = -B(f, 0) = 0$. Thus, any lift $\tilde{\chi}_T : G_0 \to \mathbb{C}^*$ of some $q\chi_T$ satisfies $\tilde{\chi}_T|_C \equiv 1$, meaning that the bundles $(E_T \otimes E_C)(G_0) \times_{\tilde{\chi}_T} \mathbb{C}^*$ and $E_T(G_0) \times_{\tilde{\chi}_T} \mathbb{C}^*$ are the same (the transition functions agree). Thus

$$\tau(E, \varphi) = \tau(E_T(G_0), e),$$

and $(E_T(G_0), e)$ is a uniformising Higgs bundle, which is maximal.

So far we have established a correspondence between $(G_0, \bar{\mathfrak{g}}_1)$-Higgs pairs with maximal Toledo invariant (equal to $-(2g-2)\,\mathrm{rank}_T(G_0, \mathfrak{g}_1)$) and K_X^m-twisted (C, V)-Higgs pairs. Now we see that it restricts to the moduli space, in other words, that if we start with a polystable $(E, \varphi) \in \mathcal{M}^{\max}(G_0, \bar{\mathfrak{g}}_1)$ the resulting (E_C, φ') is polystable. For this, pick a maximal compact $C^{\mathbb{R}} \subseteq C$ with Lie algebra $\mathfrak{c}^{\mathbb{R}} \subseteq \mathfrak{c}$. Let $s \in i\mathfrak{c}^{\mathbb{R},\Gamma}$, $P'_s \subseteq C$ the associated subgroup and $\sigma' \in H^0(X, E_C(C/P'_s))$ a reduction of structure group such that $\varphi' \in H^0(X, E_{C,\sigma'}(V_s) \otimes K_X^m)$. This element s regarded in $i\mathfrak{k}^\Gamma$ for $\mathfrak{k}$ the Lie algebra of a maximal compact K in G_0 containing $C^{\mathbb{R}}$ also defines a subgroup $P_s \subseteq G_0$, which by definition verifies $P'_s \subseteq P_s$. Thus we have a map $C/P'_s \to G_0/P_s$ which, from σ', gives a reduction $\sigma \in H^0(X, E(G_0/P_s))$. Now we need to verify that $\varphi \in H^0(X, E_\sigma(\bar{\mathfrak{g}}_{1,s}) \otimes K_X)$, in other words, that $e \in H^0(X, E_\sigma(\mathfrak{g}_{1,s}) \otimes K_X)$ and $(\mathrm{ad}(e)^{m-1})^{-1}(\varphi') \in H^0(X, E_\sigma(\mathfrak{g}_{1-m,s}) \otimes K_X)$. The former is due to the fact that $s \in i\mathfrak{c}^{\mathbb{R}} \subseteq \mathfrak{c}$, so that $[s, e] = 0$. The latter follows because (pointwise) $\mathrm{ad}(e)^{m-1}$ restricts to an isomorphism between $\mathfrak{g}_{1-m,s}$ and V_s, since for $x \in \mathfrak{g}_{1-m}$ we have $\mathrm{ad}(e)^{m-1}([s, x]) = [s, \mathrm{ad}(e)^{m-1}(x)]$ as $[s, e] = 0$. Thus, since φ' takes values in V_s, we have that φ^- takes values in $\mathfrak{g}_{1-m,s}$ as desired. Polystability of (E, φ) then gives

$$\deg E(\sigma, s) \geq 0.$$

Now, recall from before that $\mathfrak{c}$ and $\langle h \rangle$ are orthogonal via B. In particular, $B(s, h) = 0$. This means that the bundles $E_\sigma \times_{\tilde{\chi}_s} \mathbb{C}^* = ((E_T \otimes E_C)(G_0))_\sigma \times_{\tilde{\chi}_s} \mathbb{C}^*$ and $E_{C,\sigma'} \times_{\tilde{\chi}_s} \mathbb{C}^*$ are the same (or, using the differential geometric definition of the degree, that $\chi_s(F_{h_T} + F_{h_C}) = \chi_s(F_{h_C})$). Thus

$$\deg E_C(\sigma', s) = \deg E(\sigma, s) \geq 0.$$

This shows semistability, and polystability follows after checking that if $\deg E_C(\sigma', s) = 0$, by the equality of the degrees above and polystability of (E, φ) we get a reduction σ'' of E to the subgroup $L_s \subseteq P_s$. But then $E'_C := (E_{\sigma''} \otimes E_T^{-1}) \cap E_C$ is a reduction of E_C to $L'_s \subseteq P'_s$.

Thus the map restricts from $\mathcal{M}^{\max}(G_0, \bar{\mathfrak{g}}_1)$ to $\mathcal{M}_{K_X^m}(C, V)$ and it is injective due to the inverse shown above. Now we will show that if (C, V) is a Vinberg θ-pair, the map is surjective. The only thing that remains to be shown for this is that the inverse sends polystable bundles to polystable bundles. This uses the Hitchin–Kobayashi correspondence from Theorem 7. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be an involution defining a compact real form $K \subseteq G$ with $\tau(\bar{\mathfrak{g}}_i) = \bar{\mathfrak{g}}_{-i}$ and chosen so that $\tau(e) = -f$.

First, by polystability of the uniformising Higgs bundle (E_T, e) and the Hitchin–Kobayashi correspondence, there exists a metric h_T on E_T such that its curvature F_{h_T} satisfies

$$F_{h_T} + [e, -\tau_{h_T}(e)]\omega = 0$$

which means that $F_{h_T} = -[e, -f]\omega = h\omega$, that is, the curvature is constant.

On the other hand, since (E_C, φ') is polystable, there is a metric h_C on E_C such that, after fixing a metric on K_X and using it to define a metric on the power $L = K_X^m$ which defines an involution τ_{h_C}, we have

$$F_{h_C} + [\varphi', -\tau_{h_C}(\varphi')]\omega = 0,$$

for ω a Kähler form on X.

Using these we can take on $E = (E_T \otimes E_C)(G_0)$ the product metric $h_T \otimes h_C$, and its curvature is $F_{h_T} + F_{h_C}$ which satisfies:

$$F_{h_T} + F_{h_C} = h\omega - [\varphi', -\tau_{h_C}(\varphi')]\omega,$$

which can be rewritten, using the fact that $\varphi' = \mathrm{ad}^{m-1}(e)(\varphi^-)$ and that $\tau_{h_C} = \tau_{h_T \otimes h_C}$, as

$$F_{h_T} + F_{h_C} + [\mathrm{ad}^{m-1}(e)(\varphi^-), -\tau_{h_T\otimes h_C}(\mathrm{ad}^{m-1}(e)(\varphi^-))] = h\omega = -i(ih)\omega.$$

Notice that ih is central in $\mathfrak{g}_0$ (by JM-regularity) and it belongs in $\mathfrak{k}_0$ because $\tau(ih) = -i\tau([e,f]) = -i[-f,-e] = i[e,f] = ih$. Thus we can claim by the Hitchin–Kobayashi correspondence that the G_0-Higgs bundle $(E, \mathrm{ad}^{m-1}(e)(\varphi^-))$ is α-polystable for $\alpha := ih \in \mathfrak{z}_0$.

Having established this we can finally show the polystability of $(E, \varphi^+ + \varphi^-)$. Let $s \in i\mathfrak{k}_0$ and consider a reduction $\sigma \in H^0(X, E(G_0/P_s))$ with $\varphi = \varphi^+ + \varphi^- \in H^0(X, E_\sigma(\mathfrak{g}_{1,s} \oplus \mathfrak{g}_{1-m,s}) \otimes K_X)$. This means that $\varphi^+ = e \in H^0(X, E_\sigma(\mathfrak{g}_{1,s}) \otimes K_X)$ and $\varphi^- \in H^0(X, E_\sigma(\mathfrak{g}_{1-m,s}) \otimes K_X)$, so that $\mathrm{ad}^{m-1}(e)(\varphi^-) \in H^0(X, E_\sigma(V_s) \otimes K_X^m)$. The ih-polystability of $(E, \mathrm{ad}^{m-1}(e)(\varphi^-))$ then implies

$$\deg E(\sigma, s) \geq B(ih, s).$$

We show in Lemma 3 below that $B(ih, s) \geq 0$, so (E, φ) is semistable. For polystability, if $\deg E(\sigma, s) = 0$ then $B(ih, s) = 0$ and thus ih-polystability of $(E, \mathrm{ad}^{m-1}(e)(\varphi^-))$ gives a reduction σ' to a Levi $L_s \subseteq P_s$ such that $\mathrm{ad}^{m-1}(e)(\varphi^-) \in H^0(X, E_{\sigma'}(V_s^0) \otimes K_X^m)$. Using the second part of Lemma 3 we have that $e \in \mathfrak{g}_{1,s}^0$, so that $\mathrm{ad}^{m-1}(e)$ is an isomorphism between $\mathfrak{g}_{1-m,s}^0$ and V_s^0 and thus we get that $\varphi^- \in H^0(X, E_{\sigma'}(\mathfrak{g}_{1-m,s}^0) \otimes K_X)$ and hence $\varphi \in H^0(X, E_{\sigma'}(\bar{\mathfrak{g}}_{1,s}^0) \otimes K_X)$, yielding polystability of (E, φ) and completing the proof. □

It remains to establish a technical lemma used during the last part of the proof. For the case $m = 2$ this is [6, Lemma 5.6] proven using finite-dimensional affine

Geometric Invariant Theory. We provide a proof below for the general case, relying on the Hitchin–Kobayashi correspondence (at heart, it is no different from the GIT proof, but using the Hitchin–Kobayashi correspondence from Chapter 3 we avoid having to introduce new concepts).

Lemma 3. *Let $s \in i\mathfrak{k}_0$ and suppose that $e \in \mathfrak{g}_{1,s}$. Then, $B(ih, s) \geq 0$ with equality if and only if $e \in \mathfrak{g}^0_{1,s}$.*

Proof. Consider the $\mathcal{O}$-twisted $(G_0, \mathfrak{g}_1)$-Higgs pair given by the trivial principal bundle $X \times G_0$ and the constant section $e \in H^0(X, (X \times G_0)(\mathfrak{g}_1)) = H^0(X, X \times \mathfrak{g}_1)$. As the principal bundle is trivial we can choose a metric h with $F_h = 0$, as well as the constant metric on $L = \mathcal{O}$ so that

$$F_h + [e, \tau_h(e)]\omega = 0 + [e, -f]\omega = -h\omega = -i(-ih)\omega,$$

which by the Hitchin–Kobayashi correspondence of Theorem 7 means that this bundle is α-polystable for $\alpha = -ih$. Because the bundle is trivial we can take a constant reduction σ to P_s and the fact that $e \in \mathfrak{g}_{1,s}$ yields by polystability that $\deg(X \times G_0)(\sigma, s) \geq B(-ih, s)$. Since $(X \times G_0)$ is trivial, $\deg(X \times G_0)(\sigma, s) = 0$ (this is immediately seen from either of the definitions of degree) so that $0 \geq B(-ih, s)$ as desired. Equality implies by polystability that $e \in \mathfrak{g}^0_{1,s}$, and if $e \in \mathfrak{g}^0_{1,s}$ then $B(ih, s) = B(i[e, f], s) = -B(if, [e, s]) = -B(if, 0) = 0$. $\square$

Remark 14. We will shortly see that indeed the assumption that (C, V) be a Vinberg θ-pair is not automatic, but in the case $m = 2$ it always holds because (C, V) is in fact a symmetric pair, as explained in [6] or, in a more general setting, in [9, Proposition 2.6]. Similarly, for $(G_0, \mathfrak{g}_1)$-Higgs pairs (the case corresponding to $\varphi^- = 0$), we have that $V = 0$ so that (C, V) is another symmetric pair. Thus, Theorem 9 includes these already existing cases.

Example 25. We now give an example of the Cayley correspondence for $m = 3$. We will work with the Vinberg θ-pair of Example 5 corresponding to cyclic quiver representations, with ranks $(1, 1, 1)$. It is JM-regular as all the ranks are equal. In this case the isomorphism $\mathfrak{g}_{-2} = \mathbb{C} \to V$ is given by sending λ to $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$. The centralizer of $e = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is $C = \mu_3 = \left\{ \lambda \cdot Id_3 : \lambda^3 = 1 \right\}$ with Lie algebra $\mathfrak{c} = 0$. Because V is abelian we have that $\mathfrak{c} \oplus V$ is a Cartan decomposition and (C, V) a symmetric pair. Thus the Cayley correspondence applies. Given a maximal $(G_0, \bar{\mathfrak{g}}_1)$-Higgs pair $(E = L_0 \oplus L_1 \oplus L_2, \varphi)$, where $\operatorname{rank} L_j = 1$ and $\sum_{j=0}^{2} \deg L_j = 0$, the fact that $\varphi^+ = e$ means that $L_j \simeq L_0 K_X^{-j}$, so that we must have $E = K_X \oplus \mathcal{O} \oplus K_X^{-1}$. The Higgs field has to be

$$\varphi = \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\omega \in H^0(X, \mathrm{Hom}(K_X^{-1}, K_X) \otimes K_X) = H^0(X, K_X^3)$. Thus, the maximal Higgs pairs in this case are parameterized by sections of K_X^3, and the Cayley correspondence bijects each bundle into $(\mathcal{O}, \omega)$, a K_X^3-twisted (C, V)-Higgs pair.

Example 26. The previous example for ranks $(2, 2, 2)$ does not verify that (C, V) is a Vinberg θ-pair. In this case,

$$C = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix} : X \in \mathrm{SL}_2(\mathbb{C}) \right\},$$

and

$$V = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & -2X & 0 \\ 0 & 0 & X \end{pmatrix} : X \in \mathfrak{gl}_2\mathbb{C} \right\}.$$

If we select the elements $v, v' \in V$ corresponding to $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $X' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively, we get, if $B := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, that

$$[v, v'] = \begin{pmatrix} 2B & 0 & 0 \\ 0 & 8B & 0 \\ 0 & 0 & 2B \end{pmatrix} = \begin{pmatrix} 4B & 0 & 0 \\ 0 & 4B & 0 \\ 0 & 0 & 4B \end{pmatrix} + \begin{pmatrix} -2B & 0 & 0 \\ 0 & 4B & 0 \\ 0 & 0 & -2B \end{pmatrix},$$

where the first summand is in $\mathfrak{c}$ and the second in V. The fact that $[V, V]$ has nonzero projection to both V and $\mathfrak{c}$ means that (C, V) is never a Vinberg θ-pair. Thus, the Cayley correspondence injects the space of maximal $(G_0, \bar{\mathfrak{g}}_1)$-Higgs pairs into the space of (C, V)-bundles, but we do not yet know whether this map is surjective.

CHAPTER 5

The Hitchin map for cyclic Higgs bundles

5.1. The Hitchin map

An important element of the theory of Higgs pairs for Vinberg θ-pairs $(G_0, \mathfrak{g}_1)$ is the existence of a fibration of the moduli space $\mathcal{M}(G_0, \mathfrak{g}_1)$. This was intially studied by Hitchin [32] for G-Higgs bundles (which correspond to $\theta = \mathrm{Id}_G$) when G is a classical group, that is, a semisimple complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$ itself. The resulting map in that case is a fibration onto a vector space of half the dimension of the moduli space, and the generic fibre is isomorphic to an abelian variety of the same dimension as the base. For G-Higgs bundles, in terms of the symplectic structure in $\mathcal{M}(G)$ mentioned in Example 13, the fibration is a completely integrable hamiltonian system on $\mathcal{M}(G)$.

We start by giving the definition of this map, known as the **Hitchin system** (for G-Higgs bundles), **Hitchin fibration** or **Hitchin map**. By Theorem 3, the invariant ring $\mathbb{C}[\mathfrak{g}_1]^{G_0}$ is a polynomial algebra generated freely by a finite number of elements, that is, $C[\mathfrak{g}_1]^{G_0} = \mathbb{C}[f_1, \dots, f_r]$. Then, because of the G_0-invariance, it is possible, given a $(G_0, \mathfrak{g}_1)$-Higgs pair (E, φ), to evaluate the Higgs field in each of the f_i, resulting in an element $f_i(\varphi) \in H^0(X, K_X^{\deg f_i})$. The Hitchin map is defined as

$$\begin{aligned} h : \mathcal{M}(G_0, \mathfrak{g}_1) &\to \mathcal{A} := \bigoplus_{i=1}^{r} H^0(X, K_X^{\deg f_i}) \\ (E, \varphi) &\mapsto (f_1(\varphi), \dots, f_r(\varphi)) \end{aligned}$$

Example 27 (The Hitchin system)**.** In the case of Example 13, where G is a complex reductive Lie group acting on its Lie algebra $\mathfrak{g}$ by the adjoint action, already a result of Chevalley [35, Section 23.1] shows that the restriction map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{t}]^W$, where $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group, is an isomorphism, allowing that the Hitchin map be defined as explained above. The easiest case to understand is $G = \mathrm{GL}_n(\mathbb{C})$. The Lie algebra $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$ is given by endomorphisms of an n-dimensional complex vector space. Thus, given $A \in \mathfrak{gl}_n\mathbb{C}$, we can compute the characteristic polynomial:

$$p_A(x) = \det(xI_n - A) = x^n + p_1(A)x^{n-1} + \cdots + p_{n-1}(A)x + p_n(A),$$

where each $p_i \in \mathbb{C}[\mathfrak{g}]$ is a homogeneous polynomial on $\mathfrak{g}$ and, from elementary linear algebra, the value $p_i(A)$ is invariant under the action of G. In fact, we have that $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, \ldots, p_n]$. This means that if we view a $\mathrm{GL}_n(\mathbb{C})$-Higgs bundle as a rank n holomorphic vector bundle E with a section $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$ as in Example 14, the Hitchin map is given by *the characteristic polynomial of* φ and takes values in $\bigoplus_{i=1}^n H^0(X, K_X^i)$.

For the semisimple classical groups G, the situation is very similar and it is studied in depth in [32]. If $G = \mathrm{SL}_n(\mathbb{C})$, elements of the Lie algebra are traceless and thus $p_1(A) = 0$, but we still have $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_2, \ldots, p_n]$. For $G = \mathrm{Sp}_{2n}(\mathbb{C})$ and $G = \mathrm{SO}_{2n+1}(\mathbb{C})$, examining the eigenvalues of endomorphisms in the Lie algebra allows to conclude that $p_i(A) = 0$ for odd values of i, so that $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_2, p_4, \ldots, p_{2n}]$. Finally, for $G = \mathrm{SO}_{2n}(\mathbb{C})$, the same analysis applies, however the determinant also verifies $p_{2n}(A) = f^2(A)$, where f is a degree n homogeneous polynomial called the *pfaffian*. As a result, $C[\mathfrak{g}]^G = \mathbb{C}[p_2, p_4, \ldots, p_{2n-2}, f]$. In short, for classical groups the Hitchin map is given by the characteristic polynomial of the Higgs field, with the exception of $\mathrm{SO}_{2n}(\mathbb{C})$ in which a square root of the determinant has to be taken.

An interesting feature of the Hitchin system in this latter case of semisimple groups is the existence of a distinguished section, **the Hitchin section**, studied in [33]. It can easily be described in the classical case using *companion matrices*, which provide a section of the characteristic polynomial. For simplicity suppose that $G = \mathrm{SL}_n(\mathbb{C})$ and $a = (a_2, \ldots, a_n) \in \mathcal{A}$ is a point on the base of the Hitchin map, so that $a_i \in H^0(X, K_X^i)$. Take the vector bundle

$$E = K_X^{\frac{n-1}{2}} \otimes (\mathcal{O} \oplus K_X^{-1} \oplus \cdots \oplus K_X^{-n+1}),$$

where the first factor depends on the choice of a square root $K_X^{\frac{1}{2}}$ of K_X and ensures that $\det E = \mathcal{O}$. Let the Higgs field be defined by the companion matrix

$$\varphi_a = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_n \\ 1 & 0 & \ldots & 0 & -a_{n-1} \\ 0 & 1 & \ldots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

This is well defined: for example, we can identify $\mathrm{Hom}(K_X^i, K_X^{i-1})$ with $K_X^{-i} \otimes K_X^{i-1} = K_X^{-1}$, so that $\mathrm{Hom}(K_X^i, K_X^{i-1} \otimes K_X) = \mathcal{O}$ and as a consequence it makes sense to write in the above matrix $1 \in H^0(X, \mathcal{O})$. The $\mathrm{SL}_n(\mathbb{C})$-Higgs bundles (E, φ_a) are stable and $h((E, \varphi_a)) = a$.

Interestingly, this section consists entirely of $\mathrm{PSL}_n(\mathbb{R})$-Higgs bundles (cf. Examples 15, 16) and, via the non-abelian Hodge correspondence of Example 17, it defines a component of representations of the fundamental group $\pi_1(X)$ in $\mathrm{PSL}_n(\mathbb{R})$. For

$n = 2$ this is the **Teichmüller space** parameterizing complex structures on X. Consequently, the resulting component (called *Hitchin component*) for any semisimple complex G is an instance of what is known as *higher Teichmüller component*.

Remark 15. If we look at arbitrary representations $\rho : G \to \mathrm{GL}(V)$ and their corresponding pairs (G, V), the property that $\mathbb{C}[V]^G$ be a finitely generated polynomial ring (thus permitting the existence of a Hitchin map) is very rare, highlighting the importance of Vinberg θ-pairs in the theory of Higgs pairs. For a more general framework in which the invariant ring still has the desired structure, see the notion of *polar representations* studied in [16].

5.2. Spectral description for classical Higgs bundles

In the previous section we have explained how the Hitchin map, in the case of semisimple classical groups or $G = \mathrm{GL}_n(\mathbb{C})$, takes a simple form in terms of coefficients of the characteristic polynomial of the Higgs field. In this context, Hitchin [32] showed case-by-case that the generic fibre of the map is an abelian variety, by establishing a correspondence between points in the fibre and some line bundles on a finite cover of the Riemann surface X, called *spectral curve*, obtained by looking at the eigenvalues of the Higgs field in the fibre. This correspondence was also studied via algebraic geometry methods in the L-twisted case by Beauville, Narasimhan and Ramanan [4]. In this section we explain how this correspondence works.

We start by looking at $\mathrm{GL}_n(\mathbb{C})$-Higgs bundles. Recall from Example 14 that these are pairs (E, φ) where E is a rank n holomorphic vector bundle over X, and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$. We know from Example 27 that the invariant polynomials are generated by the coefficients of the characteristic polynomial $\{p_i : 1 \leq i \leq n\}$, each being homogeneous of degree i, and so the Hitchin map is

$$\begin{aligned} h : \mathcal{M}(\mathrm{GL}_n(\mathbb{C})) &\to \mathcal{A} = \bigoplus_{i=1}^{r} H^0(X, K_X^i) \\ (E, \varphi) &\mapsto (p_1(\varphi), \dots, p_n(\varphi)). \end{aligned}$$

We want to study the fibre at $a := (a_1, \dots, a_n) \in \mathcal{A}$, where each $a_i \in H^0(X, K_X^i)$ is a fixed section. This can be done using a curve $S \subseteq K_X$ which covers X via restriction of the projection $\pi : K_X \to X$ and is defined as follows. Consider the pullback

$$\begin{tikzcd} \pi^* K_X \arrow[r] \arrow[d] & K_X \arrow[d, "\pi"] \\ K_X \arrow[r, "\pi"] & X \end{tikzcd}$$

where $\pi^* K_X$ is the vector bundle on K_X that can be seen as the subset of $K_X \times K_X$ consisting of points (ω_1, ω_2) such that $\pi(\omega_1) = \pi(\omega_2)$. This has a section $\lambda : K_X \to \pi^* K_X$ given by $\omega \mapsto (\omega, \omega)$ called *tautological section*. By pulling back the sections

a_i, one also gets $\pi^*a_i \in H^0(K_X, \pi^*K_X^i)$. We can then define the **spectral curve** as

$$S_a := \{\lambda^n + \pi^*a_1\lambda^{n-1} + \cdots + \pi^*a_{n-1}\lambda + \pi^*a_n = 0\}.$$

Notice that it is defined as the vanishing locus of a section s_a of $\pi^*K_X^n$. Locally, above each point $x \in X$ the curve consists of the eigenvalues of $\varphi|_x$ for any Higgs bundle (E, φ) that projects to a via h, as S_a is defined essentially as the zero locus of the characteristic polynomial given by a.

The curve S_a is a ramified covering $\pi : S_a \to X$ of degree n which for generic a is irreducible. Moreover, by changing the values of a we get different spectral curves S_a corresponding to different divisors in K_X. Since the spectral curve is defined by zeroes of a section $s_a \in H^0(K_X, \pi^*K)$ which is a polynomial whose coefficients are given by a, the elements a and λa give the same curve. Thus we get a space of divisors parametrized by a subspace of $\mathbb{P}(H^0(K_X, \pi^*K))$, and it is in fact a linear subspace because we have that $s_{a+a'} = s_a + s_{a'}$. Such a set of curves is known as *linear system of divisors* (see e.g. [28, Chapter 1]). It is a theorem of Bertini [28, Chapter 1] that the generic curve in such a system is smooth except possibly at *base points*, which are by definition the points that belong to every curve in the system. Suppose that $p \in K_X$ is such a point. Then $p \in \{s_0 = 0\} = \{\lambda^n = 0\}$, so $\lambda(p) = 0$. But also, for all $a_n \in H^0(X, K_X^n)$ we have $p \in \{\lambda^n + \pi^*a_n = 0\}$. Thus $0 = \lambda^n(p) + \pi^*a_n(p) = 0 + \pi^*a_n(p) = \pi^*a_n(p) = a_n(\pi(p))$. This means that there is a point $\pi(p) \in X$ where every section of K_X^n vanishes, which is known to be false. We have then shown:

Remark 16. For generic values of a, the corresponding spectral curve S_a is irreducible and smooth.

It is precisely for these values that we will describe the fibre and see that it is abelian. The covering $\pi : S_a \to X$ ramifies at the points where s_a has multiple roots, so the ramification divisor R_a on S_a can be defined as the zero locus of the resultant of s_a and the section corresponding to the formal derivative $\frac{\partial s_a}{\partial \lambda}$, which is a section of $H^0(X, \pi^*K_X^{n(n-1)})$. We can then use Riemann–Hurwitz to compute the genus:

$$2g_{S_a} = n(2g-2) + \deg R_a + 2 = n(2g-2) + n(n-1)(2g-2) + 2,$$

so that $g_{S_a} = 1 + n^2(g-1)$.

Now fix $a \in \mathcal{A}$ such that $S := S_a$ is irreducible and smooth and take a line bundle $L \to S$. This gives a one dimensional vector space at each point in S which represents an eigenvalue, so it defines E and φ such that L models the eigenspaces of φ at each point. This is rigorously carried out as follows: define $E := \pi_*L$ to be the direct image bundle (that is, the bundle corresponding to the direct image sheaf of L via π). Take an open subset $U \subseteq X$ and a section $s \in H^0(\pi^{-1}(U), L)$. We then have a section $s \otimes \lambda \in H^0(\pi^{-1}(U), L \otimes \pi^*K_X)$. That is, multiplication by λ gives a map

$$H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^*K_X),$$

which, by definition of direct image, becomes a map

$$H^0(U, E) \to H^0(U, E \otimes K_X).$$

This is valid for each open set U and thus defines a vector bundle map $\varphi : E \to E \otimes K_X$, that is, a section $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$. By definition, φ has eigenvalues given by λ (we defined it as multiplication by λ on each fiber of L) and thus $\det(\lambda|_S \mathrm{Id} - \pi^*\varphi) = 0$. Since this is irreducible by the choice of a, Cayley–Hamilton shows that it is the characteristic polynomial of φ and thus $h(E, \varphi) = a$, so that we constructed a point in the fiber.

Conversely, if $(E, \varphi) \in h^{-1}(a)$, we have that $\det(\lambda - \pi^*\varphi) = s_a = 0$ on S, so that outside of R_a, where we have n distinct eigenvalues, we have a well defined decomposition into eigenspaces and these define the line bundle L. We can accomplish this rigorously via sheaf theory: the vector bundle map $\lambda - \pi^*\varphi$ can be seen as a map of locally free sheaves, and its kernel $\ker(\lambda - \pi^*\varphi) \subseteq \pi^*E$ is locally free of rank 1 by the previous observation. This defines a line bundle $L \to S$ which corresponds by the previous construction to (E, φ).

If the degree of L is d', since $\pi_*L = E$ we get that

$$\deg(E) = \deg \pi_*L = \deg L + (g-1)\deg \pi - (g_S - 1) = d' - n(n-1)(g-1).$$

We have shown:

Theorem 10 (Spectral correspondence for $\mathrm{GL}_n(\mathbb{C})$ [4, 32])**.** *For $a \in \mathcal{A}$ such that the spectral curve S_a is irreducible and smooth, there is a bijective correspondence between holomorphic line bundles over S of degree d' and points in $h^{-1}(a)$ of degree $d' + n(n-1)(g-1)$.*

Thus, the generic fiber of h is the abelian variety $\mathrm{Pic}(S_a)$ (the variety of line bundles on S_a, which is an abelian group with the tensor product), whose dimension is $g_{S_a} = n^2(g-1) + 1$, half of the dimension of the moduli space of $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles (cf. Example 18).

For each of the semisimple classical Lie groups, extra care needs to be taken in order to obtain Higgs bundles with the structural constraints corresponding to each group, as explained in Example 14. This translates in having to restrict to some subset of the line bundles on the spectral curve. We will showcase this by explaining the case of $G = \mathrm{SL}_n(\mathbb{C})$. As stated in the previous section, the lie algebra $\mathfrak{sl}_n(\mathbb{C})$ corresponds to traceless linear endomorphisms of an n-dimensional complex vector space. As such, the functions giving the Hitchin map, which still are the coefficients of the characteristic polynomial, become $\{p_2, \ldots, p_n\}$. Given $a := (a_2, \ldots, a_n) \in \mathcal{A}$ a point of the Hitchin base of $\mathrm{SL}_n(\mathbb{C})$, we define the spectral curve S_a as before, via the vanishing of the section s_a defined as a polynomial whose coefficients are given by a.

If (E, φ) is a $\mathrm{SL}_n(\mathbb{C})$-Higgs bundle, that is, E is a holomorphic vector bundle of rank n and $\det E = \mathcal{O}$, and $\varphi \in H^0(X, \mathrm{End}_0(E) \otimes K_X)$ is traceless, we can still use the spectral correspondence detailed above and get a line bundle $L \to S_a$. However, not every $L \to S_a$ gives a $\mathrm{SL}_n(\mathbb{C})$-Higgs bundle. Indeed, the obtained (E, φ) will verify that φ is traceless, because it has characteristic polynomial given by a which does not have a term in λ^{n-1}, but it could be the case that $\det E \neq \mathcal{O}$. Thus, we need to restrict to the line bundles L such that $\det(\pi_*L) = \mathcal{O}$.

This subset can be explicitly described. First, recall that for a morphism of algebraic curves $f : C \to C'$, we have a homomorphism on divisors (line bundles), known as the **norm map**:

$$\mathrm{Nm} : \mathrm{Pic}(C) \to \mathrm{Pic}(C'),$$

given by $\mathrm{Nm}(\sum_i n_i p_i) := \sum_i n_i f(p_i)$. Its kernel defines a subset of $\mathrm{Pic}(C)$ known as the **Prym variety**, $\mathrm{Prym}(C, C') := \ker \mathrm{Nm} \subseteq \mathrm{Pic}(C)$, which is a subvariety of dimension $g_C - g_{C'}$ the difference of genera.

Returning to the spectral correspondence, in [4] it is shown that for a line bundle $L \to S_a$ we have

$$\det(\pi_* L) \simeq \mathrm{Nm}(L) \otimes K_X^{-n(n-1)/2}.$$

This means that the $L \to S_a$ that we seek (those with $\det(\pi_* L) = \mathcal{O}$) are those with $\mathrm{Nm}(L) \simeq K_X^{n(n-1)/2} = \mathrm{Nm}(\pi^* K_X^{(n-1)/2})$ (for the last equality we use that $\mathrm{Nm}(\sum n_i \pi^{-1}(p_i)) = n \sum n_i p_i$ as π has degree n). Thus we want the line bundles $L \to S_a$ with

$$L \otimes \pi^* K_X^{-(n-1)/2} \in \mathrm{Prym}(S_a, X).$$

In short, the generic fibre in this case is given (up to shift by $\pi^* K_X^{-(n-1)/2}$) by the Prym variety $\mathrm{Prym}(S_a, X)$, of dimension $g_{S_a} - g = (n^2 - 1)(g - 1)$, again half the one of the moduli space. For the remaining classical semisimple Lie groups, similar arguments can be made [32] to see the generic fibres as Prym varieties for suitable curve morphisms, and it is always the case that the resulting dimension is half of the moduli space (necessary to show that h is a completely integrable system in this context).

5.3. Quasi-split cyclic Higgs bundles

Interested in the study of the Hitchin map for cyclic Higgs bundles, it is reasonable to ask whether the previous analysis can be made to describe the generic fibre and whether we can expect it to be abelian. Already in the involutive ($m = 2$) case, which was studied by Schaposnik [44, 45] using spectral curves and by García-Prada and Peón-Nieto [23, 42] using a different approach involving *cameral curves*, it can be seen that the generic fibre is no longer abelian, except for the case when the associated real form $G^{\mathbb{R}}$ is *quasi-split*.

This notion of quasi-split real form has different characterizations. Let $\theta \in \mathrm{Aut}_2(G)$ be the corresponding holomorphic involution inducing the $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\mathfrak{c} \subseteq \mathfrak{g}_1$ be a Cartan subspace as in Definition 8. We define the **regular elements** of $\mathfrak{g}_1$ for the adjoint action of G_0 as

$$\mathfrak{g}_1^{\mathrm{reg}} = \{x \in \mathfrak{g}_1 : \dim C_{G_0}(x) = \dim \mathfrak{c}\},$$

as well as those of $\mathfrak{g}$ for the adjoint action of G as

$$\mathfrak{g}^{\mathrm{reg}} = \{x \in \mathfrak{g} : \dim C_G(x) = \dim \mathfrak{t}\},$$

where $\mathfrak{t} \subseteq \mathfrak{g}$ is the corresponding Cartan subspace (in this case, subalgebra). We have the following definition for quasi-split [36, Section VI.12].

Definition 22. The real form $G^{\mathbb{R}}$ (or the involution θ) is **quasi-split** if any of the following equivalent conditions is met:

1. The centraliser $C_{\mathfrak{g}_0}(\mathfrak{c})$ is abelian.
2. There is a Borel (i.e. maximal solvable and connected) subgroup $B \subseteq G$ with a θ-invariant maximal torus $T \subseteq B$ such that $\theta(B) \cap B = T$.
3. $\mathfrak{g}_1^{\mathrm{reg}} = \mathfrak{g}_1 \cap \mathfrak{g}^{\mathrm{reg}}$.

If $C_{\mathfrak{g}_0}(\mathfrak{c}) = 0$, the real form (or involution) is called **split**.

Example 28. Consider the real form $\mathrm{SU}(p,q)$ of $\mathrm{SL}_{p+q}(\mathbb{C})$, corresponding to the involution of Example 5 where the vector spaces V_0 and V_1 have dimension p, q respectively. Without loss of generality assume that $p \leq q$. We can then split $V_1 := U_1 \oplus W_1$ where $\dim U_1 = p$, $\dim W_1 = q - p$. In Example 5 we gave a description of $\mathfrak{c}$ in terms of this splitting. Using matrices (after fixing basis which respects the splitting of V_1), as elements of $\mathfrak{sl}_{p+q}(\mathbb{C})$, we had:

$$\mathfrak{c} = \left\{ \left(\begin{array}{ccc|ccccc} 0 & \dots & 0 & \lambda_1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda_p & \dots & 0 \\ \hline \lambda_1 & \dots & 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_p & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 \end{array} \right) : \lambda_1, \dots, \lambda_p \in \mathbb{C} \right\}.$$

Its centraliser in $\mathfrak{g}_0$ is readily computed in terms of the basis used to construct $\mathfrak{c}$. With respect to the basis elements of V_0 and U_1, we need diagonal endomorphisms so that they centralize $\mathfrak{c}$. However, there are no restrictions with respect to W_1, as the elements of $\mathfrak{c}$ vanish there. Thus,

$$C_{\mathfrak{g}_0}(\mathfrak{c}) = \left\{ \left(\begin{array}{ccc|ccc|ccc} \lambda_1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_p & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda_p & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & 0 & \dots & 0 & a_{11} & \dots & a_{1(q-p)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & a_{(q-p)1} & \dots & a_{(q-p)(q-p)} \end{array} \right) : \lambda_1, \dots, \lambda_p, a_{ij} \in \mathbb{C} \right\}_0.$$

The subscript 0 at the very end means considering only traceless elements. This centralizer is abelian if and only if each element belonging to it is diagonal, hence if and only if $q-p \in \{0,1\}$. Thus, the only quasi-split cases of this example are $\mathrm{SU}(p,p)$ and $\mathrm{SU}(p,p+1)$. Notice that there is always one of these occuring in each $\mathrm{SL}_n(\mathbb{C})$ (this is a general fact: quasi-split forms always exist). The only split case corresponds to $p=q=1$, giving $\mathrm{SU}(1,1) \simeq \mathrm{SL}_2(\mathbb{R})$. (In general, $\mathrm{SL}_n(\mathbb{R}) \subseteq \mathrm{SL}_n(\mathbb{C})$ is always split).

In general, for G semisimple, the quasi-split forms are classified: they are the split ones, as well as those with Lie algebra isomorphic to $\mathfrak{su}(p,p)$, $\mathfrak{su}(p,p+1)$, $\mathfrak{so}(p,p+2)$ and $\mathfrak{e}_6(2)$.

In general, for cyclic Higgs bundles, we want to give a general definition of a quasi-split Vinberg θ-pair (or quasi-split order m automorphism θ). A reasonable definition, given the fact that we already have all the necessary elements and they are related to the definition of the Hitchin map, is

Definition 23. Let $(G_0,\mathfrak{g}_1)$ be a Vinberg θ-pair, for $\theta \in \mathrm{Aut}_m(G)$. Let $\mathfrak{c} \subseteq \mathfrak{g}_1$ be a Cartan subspace. We say that the pair (or the automorphism θ) is **quasi-split** if the centralizer $C_{\mathfrak{g}_0}(\mathfrak{c})$ is abelian. We say that it is **split** if $C_{\mathfrak{g}_0}(\mathfrak{c}) = 0$.

An interesting question that we hope to address in the future is whether the other two characterizations for quasi-splitness in the case of real forms are still applicable here. We conclude this section by classifying the quasi-split Vinberg θ-pairs of the cyclic quiver case from Example 5.

Proposition 11. *Let $(G_0,\mathfrak{g}_1)$ be the Vinberg θ-pair of $G=\mathrm{SL}_n(\mathbb{C})$ from Example 5, coming from the space of representations of a cyclic quiver where the dimension of the i-th vertex is d_i. Then, it is quasi-split if and only if for some integer $k \geq 1$ and for all $i \in \{0,\dots,m-1\}$ we have $d_i \in \{k,k+1\}$, and it is split if and only if for all $i \in \{0,\dots,m-1\}$ we have $d_i = 1$.*

Proof. We start recalling from Example 5 the description of the Cartan subspace $\mathfrak{c}$. Assume without loss of generality that d_0 is minimal (otherwise, rotate the pieces V_j). Fix a splitting $V_j = U_j \oplus W_j$ of each vector space such that $\dim U_j = d_0$, and a basis $B_j = \{v_1^j,\dots,v_{d_0}^j\}$ for each U_j. Consider, for $\lambda = (\lambda_1,\dots,\lambda_{d_0}) \in \mathbb{C}^{d_0}$, the element $f^\lambda = (f_0^\lambda,\dots,f_{m-1}^\lambda) \in \mathfrak{g}_1$ defined by $f_j^\lambda(v_k^j) := \lambda_k v_k^{j+1}$, $f_j^\lambda|_{W_j} \equiv 0$. Then, $\mathfrak{c} := \{f^\lambda : \lambda \in \mathbb{C}^m\}$. From this description, as in the $m=2$ case shown in Example 28 above, elements in the centralizer need to be diagonal at each U_j and are unrestricted at the W_j (because elements of $\mathfrak{c}$ vanish at the W_j). That is, the centralizer consists of elements of the form $g^\mu = (g_0^\mu,\dots,g_{m-1}^\mu) \in \mathfrak{g}_0$, where $\mu = (\mu_0,\dots,\mu_{m-1}) \in \mathbb{C}^{d_0}$, and each $g_j^\mu \in \mathrm{End}(V_j)$ is such that $g_j^\mu(v_k^j) := \mu_k v_k^{j+1}$ and without extra restrictions for $g_j^\mu|_{W_j}$ (because of this, each μ gives many elements). We also require that $g^\mu \in \mathrm{End}(\bigoplus_j V_j)$ is traceless. In short:

$$C_{\mathfrak{g}_0}(\mathfrak{c}) = (\mathbb{C}^{d_0} \oplus \bigoplus_{j=1}^{m-1} \mathfrak{gl}_{d_i-d_0}(\mathbb{C}))_0.$$

The subscript 0 means taking the traceless elements. This is abelian if and only if $d_i - d_0 \in \{0,1\}$ for all i, and is zero if and only if $d_i = 1$ for all i. $\square$

5.4. Spectral data for type $(k, k, \ldots, k)$ cyclic $\mathrm{SL}_{mk}(\mathbb{C})$-Higgs bundles

As established in the previous section, one of the examples of quasi-split Vinberg θ-pairs occurs in the case of cyclic quiver representations of Example 5 when all the dimensions are the same: $d_0 = \cdots = d_{m-1} = k \geq 1$. This is a Vinberg θ-pair $(G_0, \mathfrak{g}_1)$ for $\mathrm{SL}_{mk}(\mathbb{C})$. In this section we will extend the results of Schaposnik [45] for the case $m = 2$ to show that a spectral description can be done as well. Using this description we will also be able to recover the Arakelov–Milnor–Wood inequality of Theorem 8 for this case.

Recall that these cyclic Higgs bundles are of the form (E, φ) where $E = W_0 \oplus \cdots \oplus W_{m-1}$ is a vector bundle with trivial determinant that splits as the direct sum of rank k vector bundles, and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$ verifies $\varphi(W_j) \subseteq W_{j+1} \otimes K_X$ for $j \in \mathbb{Z}/m\mathbb{Z}$. As explained in Section 3.3, these are fixed points of the automorphism $(E, \varphi) \mapsto (E, \zeta\varphi)$ where ζ is a primitive m-th root of unity. This is a consequence of the fact that every $A \in \mathfrak{g}_1$ is G_0-conjugate to ζA. In particular this implies that the characteristic polynomial of A is of the form

$$\det(x\,\mathrm{Id} - A) = x^{mk} + p_1(A)x^{mk-m} + \cdots + p_{k-1}(A)x^m + p_k(A).$$

That is, a polynomial in x^m. The homogeneous polynomials $p_1, \ldots, p_k$ generate the ring of invariants $\mathbb{C}[\mathfrak{g}_1]^{G_0}$, as can be deduced from the explanation at the end of Example 5. Thus the Hitchin map is again given by the coefficients of the characteristic polynomial.

Fix a point $a := (a_1, \ldots, a_k) \in \mathcal{A}$, where $a_i \in H^0(X, K_X^{mi})$. Define the spectral curve $S_a \subseteq K_X$ as in Section 5.2 and assume that it is irreducible and smooth. Recall that its genus is $g_{S_a} = (mk)^2(g-1)+1$. Given a point $(E, \varphi) \in h^{-1}(a)$, the spectral correspondence from Section 5.2 gives a line bundle $L \to S_a$ in the Prym variety $\mathrm{Prym}(S_a, X)$. However, as it happened when specializing the correspondence from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{SL}_n(\mathbb{C})$, it will not be true in general that a line bundle $L \in \mathrm{Prym}(S_a, X)$ produces a cyclic Higgs bundle of type $(k, k, \ldots, k)$ through the spectral correspondence. In order to identify those who do, we first notice that the spectral curve $S := S_a$ is endowed with an order m automorphism

$$\begin{aligned} \sigma : S &\to S \\ \lambda &\mapsto \zeta\lambda. \end{aligned}$$

That is, the automorphism which over a point of x permutes the eigenvalues via multiplication by ζ. This is well defined as the section defining S is a polynomial on λ^m. Then, we have:

Proposition 12. *The line bundle $L \in \mathrm{Prym}(S, X)$ defines via the spectral correspondence a cyclic $\mathrm{SL}_{mk}(\mathbb{C})$-Higgs bundle of type $(k, k, \ldots, k)$ if and only if $\sigma^* L \simeq L$.*

Proof. Suppose that $\sigma^* L \simeq L$. We have the diagram

$$\begin{tikzcd} L \arrow[r, "\simeq"] \arrow[d] \arrow[rr, bend left, "\bar{\sigma}"] & \sigma^* L \arrow[r] \arrow[d] & L \arrow[d] \\ S \arrow[r, "\mathrm{Id}"] & S \arrow[r, "\sigma"] & S \end{tikzcd},$$

which lifts σ to an automorphism $\bar{\sigma}$ of L, that is, gives a linearisation of the action of $\mathbb{Z}/m\mathbb{Z}$ on S to L. We will abuse notation and denote $\sigma \in \mathrm{Aut}(L)$ in what follows. Denote by $\pi : S \to X$ the ramified covering. Note that $\pi \circ \sigma = \pi$, so that if $U \subset X$ is open then $\pi^{-1}(U) \subseteq S$ is σ-invariant and open. This means that σ acts on the local sections $H^0(\pi^{-1}(U), L)$ (for a section s we set $(\sigma s)(p) := \sigma(s(\sigma^{-1}(p))))$. This is a linear map of order m on a complex vector space and hence we can decompose

$$H^0(\pi^{-1}(U), L) = \bigoplus_{j=0}^{m-1} H^0(\pi^{-1}(U), L)^j,$$

where $H^0(\pi^{-1}(U), L)^j$ is the subspace where σ acts via multiplication by ζ^j. Set $E = \pi_* L$ the associated vector bundle via the spectral correspondence. The definition of direct image yields $H^0(\pi^{-1}(U), L) = H^0(U, E)$, so we obtain a decomposition

$$H^0(U, E) = \bigoplus_{j=0}^{m-1} H^0(U, E)^j.$$

As the open subset $U \subseteq X$ is arbitrary, this induces a vector bundle decomposition $E = \bigoplus_{j=0}^{m-1} W_j$. Now we see that each piece has rank k. Select a point $x \in X$ outside of the ramification locus of $\pi : S \to X$ and with $a_k(x) \neq 0$, so that $\pi^{-1}(x) \subseteq S$ contains mk distinct nonzero eigenvalues which can be split into k groups of m which are cyclically permuted by σ (the nonzero assumption is necessary so that they are not fixed by σ). In other words,

$$\pi^{-1}(x) = \{e_1, \dots, e_k, \sigma(e_1), \dots, \sigma(e_k), \dots, \sigma^{m-1}(e_1), \dots, \sigma^{m-1}(e_k)\}.$$

By looking at the corresponding eigenvectors in that order, we have that $E|_x = \bigoplus_{j=0}^{m-1} \mathbb{C}^k$ and σ acts on $E|_x$ by the cyclic permutation

$$(v_0, \dots, v_{m-1}) \mapsto (v_{m-1}, v_0, \dots, v_{m-1}).$$

Since $W_j|_x$ is the subspace where σ acts via multiplication by ζ^j, we can write it explicitly as $\{(v, \zeta^{j(m-1)}v, \zeta^{j(m-2)}v, \dots, \zeta^j v) : v \in \mathbb{C}^k\}$, thus it is k-dimensional as required. Recall as well that, via the spectral correspondence, φ is retrieved by the direct image of the map given by multiplying by λ on sections of L. Since $\sigma(\lambda) = \zeta\lambda$, sections where σ acts by ζ^j are sent to those where it acts as $\zeta^j \zeta = \zeta^{j+1}$. Through the direct image this means $\varphi(W_j) \subseteq W_{j+1} \otimes K_X$, as desired.

Conversely, if (E, φ) is cyclic, we have that $(E, \varphi) \simeq (E, \zeta\varphi)$, implying that the line bundle $L \to S$ of eigenspaces obtained by spectral correspondence has an induced automorphism over σ in S (that is, the induced automorphism sends the eigenspace for λ to that of $\zeta\lambda$). This means, with a diagram as in the start of the proof, that $\sigma^* L \simeq L$. □

Thus, the spectral correspondence bijects the locus in $\mathrm{Prym}(S, X)$ of line bundles L that satisfy the additional condition $\sigma^* L \simeq L$ with the cyclic $\mathrm{SL}_{mk}(\mathbb{C})$-Higgs bundles of type $(k, \ldots, k)$. Now we will see how the degrees of each piece W_j can be recovered, and we will use it to derive the bound on the Toledo invariant. It will be convenient to introduce the quotient curve $\bar{\pi} : \bar{S} := S/\sigma \to X$, a degree k ramified cover of X inside K_X^m, given by the equation

$$\xi^k + a_1 \xi^{k-1} + \cdots + a_{k-1}\xi + a_k = 0,$$

where $\xi := \lambda^m$. The quotient $\rho : S \to \bar{S}$ is a covering of degree m, which ramifies only when $\lambda = 0$ (in this case, the fibre has a single point). In S, λ vanishes precisely when $a_k \in H^0(X, K_X^{mk})$ does, so there are $\deg K_X^{mk} = 2mk(g-1)$ such points. The Riemann–Hurwitz formula then yields the genus of the quotient curve

$$g_{\bar{S}} = 1 + (g-1)(mk^2 - (m-1)k).$$

As before, if $U \subseteq \bar{S}$ is an open subset of the quotient curve, then $\rho^{-1}(U)$ is σ-invariant and if $L \to S$ is a line bundle with $\sigma^* L \simeq L$ we can decompose

$$H^0(\rho^{-1}(U), L) = \bigoplus_{j=0}^{m-1} H^0(\rho^{-1}(U), L)^j.$$

Thus,

$$H^0(U, \rho_* L) = \bigoplus_{j=0}^{m-1} H^0(U, \rho_* L)^j,$$

and we get a decomposition $\rho_* L = \bigoplus_{j=0}^{m-1} L_j$ where each $L_j \to \bar{S}$ is a line bundle, and by using the fact that $\pi = \bar{\pi} \circ \rho$, we have $\bar{\pi}_* L_j = W_j$.

The degrees of each W_j can be recovered by studying the points $p \in S$ that are fixed by σ, that is, those with $\lambda(p) = 0$ which occur when $a_k(p) = 0$. This shows that there are $\deg(K_X^{km}) = 2km(g-1)$ such points. When p is fixed by σ, the linearisation on L gives an automorphism of order m of the one-dimensional fibre $L|_p$, thus it is given by multiplication by ζ^j for some $j \in \{0, \ldots, m-1\}$. Let M_j be the number of fixed points p where σ acts by ζ^j on $L|_p$. We have $\sum_{j=0}^{m-1} M_j = 2mk(g-1)$.

We will need to use an auxiliary line bundle $L' \to \bar{S}$ of large degree. This gives the line bundle $\rho^* L' \to S$ and, since $\rho \circ \sigma = \rho$, we get $\sigma^*(\rho^* L') = (\rho \circ \sigma)^* L' = \rho^* L'$, so $\rho^* L'$ is also fixed by σ^* and, as in the proof of the previous proposition, σ acts on $\rho^* L'$ over $\sigma : S \to S$. Since $\rho^* L'|_p = L'|_{\rho(p)}$ and the action of σ on $\bar{S}$ is the identity (because $\lambda \mapsto \zeta\lambda$ becomes $\xi = \lambda^m \mapsto (\zeta\lambda)^m = \lambda^m = \xi$), we get that σ acts as the identity on $\rho^* L'|_p$. In short, we can consider the line bundle $L \otimes \rho^* L'$ instead of L, which is still σ^*-fixed and is such that σ acts via multiplication by ζ^j on the fibres of the same M_j fixed points where it did on L, while also having the advantage of its degree being large.

If L' has degree large enough for $H^1(S, L\otimes\rho^* L') = 0$, and we set $d_j := \dim H^0(S, L\otimes \rho^* L')^j$ (this subspace defined exactly in the same way as for L), Riemann–Roch gives

$$\sum_{j=0}^{m-1} d_j = \dim H^0(S, L\otimes\rho^* L') = \deg L + m\deg L' - m^2k^2(g-1). \tag{5.1}$$

In order to get additional equations, we will need the following result which is a weaker version of [2, Theorem 4.12].

Proposition 13. *Let X be a Riemann surface, $F \to X$ a holomorphic vector bundle, $f : X \to X$ a holomorphic map with finitely many fixed points $P \subseteq X$ and $\bar f : F \to F$ a holomorphic bundle homomorphism over f. Suppose that $H^1(X, F) = 0$. Let $H^0(f, \bar f)$ denote the induced linear map on $H^0(X, F)$. Then*

$$\operatorname{tr} H^0(f, \bar f) = \sum_{p\in P} \frac{\operatorname{tr} F|_p}{1 - df|_p}.$$

We apply this for the Riemann surface S, the vector bundle $L\otimes\rho^* L'$, the maps $\sigma^i : S \to S$ and the corresponding lifts $\sigma^i : L\otimes\rho^* L' \to L\otimes\rho^* L'$ for $i \in \{1, \dots, m-1\}$. Each σ^i acts via multiplication by ζ^{ij} on $H^0(S, L\otimes\rho^* L')^j$, so that

$$\operatorname{tr} H^0(\sigma, \sigma^i) = \sum_{j=0}^{m-1} \zeta^{ij} d_j.$$

On the other hand, as σ^i is multiplication by ζ^i on S, we get $df|_p = \zeta^i$. As $\sigma^i|_p = \zeta^{ij}\,\mathrm{Id}$ on the fibres of exactly M_j fixed points of σ^i, we can conclude by Proposition 13 that

$$\sum_{j=0}^{m-1} \zeta^{ij} d_j = \sum_{j=0}^{m-1} \frac{\zeta^{ij} M_j}{1-\zeta^i} \qquad i \in \{1, \dots, m-1\}. \tag{5.2}$$

Using the $m-1$ equations in (5.2) together with the extra equation (5.1), we have a linear system of m equations and m variables. Solving for d_j requires multiplying each of the $m-1$ equations in (5.2) by ζ^{-ij} and summing the m resulting equations. The coefficient in d_k is then $\sum_{i=0}^{m-1} \zeta^{-ij}\zeta^{ik} = \sum_{i=0}^{m-1} \zeta^{i(k-j)}$, which is m if $k = j$ and 0 otherwise. This results in

$$d_j = \deg L' + \frac{1}{m}\left(\deg L - m^2k^2(g-1) + \sum_{l=0}^{m-1} \alpha_{j,l} M_l\right) \tag{5.3}$$

with

$$\alpha_{j,l} := \sum_{r=1}^{m-1} \frac{\zeta^{r(l-j)}}{1-\zeta^r} = \overline{(l-j)} - 1 - \frac{m-1}{2}, \tag{5.4}$$

where for an integer $b \in \mathbb{Z}$ the notation $\bar{b}$ is the remainder of dividing b by m such that $1 \leq \bar{b} \leq m$. This latter identity can be checked either via analytical (using residues) or algebraic (using polynomial identities) methods.

Now we remove the dependency on the auxiliary line bundle L'. Recall the line bundles $L_j \to \bar{S}$ from before, and note that $H^0(S, L \otimes \rho^* L')^j = H^0(\bar{S}, L_j \otimes L')$ by taking direct image via ρ. By Riemann–Roch (recall that the degree of L' is large so that $H^1(\bar{S}, L_j \otimes L') = 0$) we obtain

$$d_j = \dim H^0(S, L \otimes \rho^* L')^j = \dim H^0(\bar{S}, L_j \otimes L') =$$
$$= \deg L_j + \deg L' - (g-1)(mk^2 - (m-1)k).$$

Using the above identity together with (5.3), we get

$$\deg L_j = (g-1)(mk^2 - (m-1)k) + \frac{1}{m}\left(\deg L - m^2k^2(g-1) + \sum_{l=0}^{m-1} \alpha_{j,l} M_l\right),$$

which means that

$$\deg W_j = \deg \bar{\pi}_* L_j = \deg L_j + k(g-1) - (g-1)(mk^2 - (m-1)k) =$$
$$= \frac{1}{m}\left(\deg L - m^2k^2(g-1) + \sum_{l=0}^{m-1} \alpha_{j,l} M_l\right) + k(g-1).$$

We have thus proven the following.

Proposition 14. *Let $L \to S$ be a line bundle with $\sigma^* L \simeq L$ and consider the induced type $(k,k,\dots,k)$ cyclic $\mathrm{SL}_{mk}(\mathbb{C})$-Higgs bundle (E,φ) via the spectral correspondence, with $E = \bigoplus_{j=0}^{m-1} W_j$ the splitting into pieces of rank k with $\varphi(W_j) \subseteq W_{j+1} \otimes K_X$ for all $j \in \mathbb{Z}/m\mathbb{Z}$. Let M_j be the number of σ-fixed points $p \in S$ such that the action of σ on $L|_p$ is via multiplication by ζ^j. Let $\alpha_{j,l}$ as in (5.4). Then, we have*

$$\deg W_j = \frac{1}{m}\left(\deg L - m^2k^2(g-1) + \sum_{l=0}^{m-1} \alpha_{j,l} M_l\right) + k(g-1).$$

As an application of this, we can derive the bound on the Toledo invariant from Theorem 8, as well as show the existence of an upper bound for this case. The bound we will obtain is the coarse one that uses $\mathrm{rank}_T(G_0, \mathfrak{g}_1)$ computed in Example 23. Keeping the notation from previous proposition, recall from Example 21 that the Toledo invariant in this case is given by

$$\tau = 2\sum_{j=0}^{m-1}\left(j - \frac{m-1}{2}\right)\deg W_j.$$

Since $\sum_{j=0}^{m-1}\left(j - \frac{m-1}{2}\right) = 0$, after substituting the expression for the degrees given in Proposition 14, we get

$$\tau = \frac{2}{m}\sum_{j=0}^{m-1}\left(j - \frac{m-1}{2}\right)\sum_{l=0}^{m-1}\alpha_{j,l}M_l = \frac{2}{m}\sum_{l=0}^{m-1} c_l M_l,$$

where $c_l := \sum_{j=0}^{m-1} \left(j - \frac{m-1}{2}\right) \alpha_{j,l}$. Using the closed expression for $\alpha_{j,l}$ given in (5.4), we get

$$c_l = -\frac{1}{2}l^2 m + \frac{1}{2}lm^2 - \frac{1}{12}m^3 + \frac{1}{12}m.$$

The concave parabola defined by the quadratic equation on l given above has its vertex at $\frac{m}{2}$. This means that $C := \max_{l\in\{0,\dots,m-1\}}\{|c_l|\}$ is bounded above by the maximum of the absolute values of said parabola at 0, $m-1$ and $\frac{m}{2}$, which are $\frac{m(m-1)(m+1)}{12}$, $\left|\frac{m(m-1)(m-15)}{12}\right|$ and $\frac{m(m^2+2)}{24}$, respectively. Since $m \geq 2$, the largest of these values is the second one and hence $C \leq \frac{m(m-1)(m+1)}{12}$. This finally yields

$$|\tau| = \frac{2}{m}\left|\sum_{l=0}^{m-1} c_l M_l\right| \leq \frac{2}{m}\sum_{l=0}^{m-1}|c_l|M_l \leq \frac{2C}{m}\cdot\sum_{l=0}^{m-1} M_l \leq$$

$$\leq \frac{2}{m}\cdot\frac{m(m-1)(m+1)}{12}\cdot 2km(g-1) = \frac{km(m-1)(m+1)}{6}(2g-2),$$

which is the coarse Arakelov–Milnor–Wood inequality from Theorem 8 computed in Example 23, together with a symmetric upper bound. The latter is not generally attained, in fact, it can be refined by bounding $\max_{l\in\{0,\dots,m-1\}}\{c_l\}$ (without the absolute values) by the value $\frac{m(m^2+2)}{24}$ of the parabola at the vertex, which matches the one above for $m = 2$ but is tighter for $m > 2$.

CHAPTER 6
Conclusions

We have studied several aspects of the moduli spaces of θ-cyclic G-Higgs bundles, which we defined as the image in $\mathcal{M}(G)$ of Higgs pairs associated to a Vinberg θ-pair, a Lie theoretical object arising from a finite order automorphism of a complex Lie group G. The study has taken two main directions. The first one has been about a selected topological invariant on the resulting moduli space, the Toledo invariant, which we have defined extending previously existing cases to our general context. This extended definition is relevant, as we have evidenced providing a bound that holds in the moduli spaces of cyclic Higgs bundles that reveals some rigidity phenomenon when it is attained. We have explored this latter phenomenon as well, revealing that the locus of cyclic Higgs bundles attaining the bound injects into the moduli space of Higgs pairs for a smaller subgroup $C \leq G$. We also provided a condition for the map to be surjective.

The other direction has focused on the Hitchin map, a fibration that exists in the moduli space of cyclic Higgs bundles onto an affine base. For standard G-Higgs bundles its generic fibre is abelian, but for cyclic Higgs bundles this need not be true, and we introduced the notion of *quasi-split* Vinberg θ-pairs as the natural candidates for the fibre to be abelian. Then we explored a particular case of quasi-split Vinberg θ-pair for the group $G = \mathrm{SL}_n(\mathbb{C})$, in which we used spectral techniques to describe the fibres and verify their abelianness.

There are still plenty of open questions and future directions that can be pursued. Here we collect some:

- In the previously existing situations where a Toledo invariant was defined, namely $G^{\mathbb{R}}$-Higgs bundles for a real form $G^{\mathbb{R}}$ of hermitian type and fixed points of the $\mathbb{C}^*$-action, the Toledo invariant has an upper bound. In our proof for the lower bound in Theorem 8, we saw that the proof breaks in the general case. However, at the end of Section 5.4 we have seen examples of moduli spaces of cyclic Higgs bundles which are not in the aforementioned two classes yet the Toledo invariant has an upper bound. It would be interesting to understand in which situations the Toledo invariant is bounded above.
- We have seen that for the locus of maximal cyclic Higgs bundles there exists a Cayley correspondence that injects it into another moduli space, and we have

shown that in some cases the map is bijective. However we have not yet found any counterexample where the map is not surjective. Possible future work is to study the surjectiveness of the map using diferent techniques than the ones used here, namely the Hitchin–Kobayashi correspondence that fails in some cases.

- We have defined quasi-split Vinberg θ-pairs in terms of one of the many equivalent definitions for quasi-splitness existing for real forms (cf. Section 5.2). We are interested in what would be the natural extension for the other equivalent definitions to cyclic Higgs bundles, if any.

- The description of the fibres of the Hitchin fibration for the remaining quasi-split pairs of inner type in $G = \mathrm{SL}_n(\mathbb{C})$, namely the K_X-twisted cyclic quiver representations of type $(k+a_1, k+a_2, \ldots, k+a_m)$ where $a_i \in \{0, 1\}$. An approach for this is to notice that the spectral curve has a component corresponding to $\lambda^r = 0$ where r is the number of a_i that are 1. For the involutive case $(m = 2)$ this means that one of the two factor subbundles has a 1-dimensional $(r = 1)$ kernel which can be quotiented out to reduce to the (k, k) case, but in general this approach is more subtle.

- A general study for the Hitchin fibration on the moduli space of cyclic Higgs bundles, first for quasi-split Vinberg pairs and then in more generality. Here we have restricted to the case of cyclic quiver representations corresponding to a quasi-split pair of $G = \mathrm{SL}_n(\mathbb{C})$. The more general case could involve more machinery that has been used for $m = 2$, such as cameral curves [23] or regular quotients [29].

- We have seen that the Hitchin map for G-Higgs bundles has a section, the Hitchin section, whose image produces a connected component of the moduli space with many interesting properties. For the involutive case of $m = 2$, there also exists a section known as Hitchin–Kostant–Rallis section [24]. An important future direction is the construction of a section of the Hitchin map, relying on the fact that the quotient map $\mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_0$ has a section, known as *Kostant–Weierstrass section*.

- If G is complex semisimple, the only examples of θ-cyclic G-Higgs bundles observed so far inside of the Hitchin section for G-Higgs bundles happen to be quasi-split. It would be interesting to see to what extent this holds.

- As mentioned in the introduction, cyclic Higgs bundles have appeared in the literature in different contexts. It would be nice to explore how the aspects introduced here (such as the Toledo invariant) can be interpreted in those pictures. For example, Baraglia [3] identified that the solutions to *affine Toda equations* on the Hitchin component were precisely the cyclic Higgs bundles therein. This allows to conclude that a (naturally associated to a Higgs bundle) harmonic map from the universal cover $\tilde{X}$ of the Riemann surface to the symmetric space $G^{\mathbb{R}}/K$ where $G^{\mathbb{R}} \leq G$ is the split real form and $K \leq G^{\mathbb{R}}$ is a maximal compact, lifts in the case of cyclic Higgs bundles in the Hitchin section to a harmonic map $\tau' : \tilde{X} \to G^{\mathbb{R}}/K'$ for a smaller subgroup $K' \leq K$. It would be interesting to study this phenomenon in general for our notion of cyclic Higgs bundles (which exists outside of the Hitchin section).

- When $m = 2$, we have seen in Example 4 that any Vinberg pair is related to a real form $G^{\mathbb{R}} \subseteq G$ and the resulting moduli space $\mathcal{M}(G^{\mathbb{R}})$ is particularly interesting because of its relation with the theory of representations of $\pi_1(X)$ in $G^{\mathbb{R}}$. An interesting question would be to study θ-cyclic Higgs bundles inside of $\mathcal{M}(G^{\mathbb{R}})$, corresponding to fixed points of the action of finite cyclic groups in $\mathcal{M}(G^{\mathbb{R}})$. If the real form is associated to an involution $\theta' \in \mathrm{Aut}_2(G)$, the problem is then related to the study of the finite order automorphisms $\theta \in \mathrm{Aut}_m(G)$ such that θ and θ' commute. In terms of gradings of $\mathfrak{g}$, this is nothing but a $(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-grading of $\mathfrak{g}$, which for odd values of m we know how to classify (as they agree with $\mathbb{Z}/2m\mathbb{Z}$-gradings) but for even values of m the classification is more complicated.

- We have shown that the Toledo invariant is bounded and a that a Cayley correspondence happens, using the fact that the corresponding Vinberg θ-pair is special. It would be interesting to study the Lie theory and invariant theory for these kind of special pairs more in detail. For example, giving additional properties of the Cartan subspaces or the Kostant–Weierstrass section in the special case may help with the study of the Hitchin section in this case.

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