Hitchin map in the minuscule case of GL_n and equiv. Cohomology.

(Miguel González) Bonn-Vienna block seminar: equiv.coh, stable envelopes and big algebras. (Feb. 2025).

Goal: Show how equiv. coh. appears in the study of the Hitchin map for GL_n-Higgs bundles (Hausel-Hichin, 2022)

$C \rightsquigarrow$ smooth projective complex curve $g \geq 2$, $K_C = T^*C$

Defn. A (GL_n-) Higgs bundle is a pair (E, φ) where

- E is a vector bundle over C of rank n, deg d
- $\varphi \in H^0(End(E) \otimes K_C)$ i.e $\varphi : E \to E K_C$

Example. $E = \mathcal{O}_C \oplus K_C^{-1} \oplus \cdots \oplus K_C^{-n+1}$

$$\varphi_0 = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & 1 & \ddots & \\ & & \ddots & \\ & & 1 & 0 \end{pmatrix} \qquad 1 : K_C^{-i} \to K_C^{-i-1} \otimes K_C .$$

Want to define a moduli space: A Higgs bundle is (semi-)stable if every subbundle $V \subsetneq E$, $V \neq 0$, $\varphi(V) \subseteq V K_C^{-1}$ has

$$\frac{\deg V}{\operatorname{rk} V} \underset{(-)}{<} \frac{d}{n}$$

Defn. The moduli space of Higgs bundles

$\mathcal{M}(n,d)$ is a q-proj. variety parameterising semistable Higgs bundles $\rightarrow$ smooth locus $\mathcal{M}^s \subseteq \mathcal{M}(n,d)$ stable Higgs bundles.

It has a canonical symplectic structure ω.
($T_{(E,\varphi)} \cong \mathbb{H}^1(\operatorname{End} E \to \operatorname{End} E \otimes K)$, project to $H^1(\operatorname{End} E)$ and Serre pair with φ)

Defn. The Hitchin map is a proper, completely integrable system given by the char. poly:

$$h : \mathcal{M}(n,d) \longrightarrow \bigoplus_{i=1}^{n} H^i(K_C^i) =: \mathcal{A}$$

$$(E,\varphi) \longmapsto (a_1, \dots, a_n)$$

s.t $\operatorname{charpoly}(\varphi) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n$

In our previous example, $h(E,\varphi_0) = 0$. We can extend to a section.

Note that h is just a global version of

$$\chi : \mathfrak{gl}_n \longrightarrow \mathfrak{gl}_n /\!\!/ GL_n = \operatorname{Spec} \mathbb{C}[a_1, \dots, a_n]$$

So we can use the Kostant section.

We see $e = \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix} \in \mathfrak{gl}_n$ as a regular nilp.

$\rightsquigarrow \mathfrak{sl}_2$-triple (e, h, f). $h = \begin{pmatrix} n-1 & & \\ & \ddots & \\ & & 1-n \end{pmatrix}$

Defn. Kostant section is $S := e + C_{\mathfrak{g}}(f) = e + \langle f_1, \dots, f_n \rangle$

where f_i are the lowest weight vectors of $\mathfrak{sl}_2 \curvearrowright \mathfrak{g}$.

Then set $\varphi_a = e + a_1 f_1 + \cdots + a_n f_n$

. The bundle E has been constructed so that $\varphi_a \in H^0(\mathrm{End}\,E \otimes K_C)$. f_i's can be scaled for $h(E, \varphi_a) = (a_1, \dots, a_n)$.

$$E = \mathcal{O} \oplus K_C^{-1} \oplus K_C^{-2}$$

$$\varphi_a = \begin{pmatrix} a_1/3 & \frac{a_1^2}{6} + \frac{a_2}{2} & -\frac{4a_1^3}{27} - \frac{a_1 a_2}{3} - a_3 \\ 1 & a_1/3 & \frac{a_1^2}{6} + \frac{a_2}{2} \\ & 1 & a_1/3 \end{pmatrix}$$

Defn. (E, φ_a) is called the <u>Hitchin section</u>.

It defines a closed Lagrangian subvariety

$W_0^+ \subseteq \mathcal{M}(r,d)$ and $h|_{W_0^+}$ is 1-1.

Goal: Other Lagrangian subvarieties? (more sophisticated)

<u>Hecke transformations</u> (minuscule).

Fix $c \in C$ and let $V \subseteq E|_c$ with $\varphi_c(V) \subseteq V$.

Defn. $\mathcal{H}_V(E, \varphi) = (E', \varphi')$ with

$$\begin{array}{ccccccccc} 0 & \to & E' & \hookrightarrow & E & \to & E|_c/V \otimes \mathcal{O}_c & \to & 0 \\ & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \bar{\varphi} & & \\ 0 & \to & E'K_c & \hookrightarrow & EK_c & \to & E|_c/V \otimes K_c & \to & 0 \end{array}$$

New Lagrangians $W_k^+ := \{\mathcal{H}_V(E, \varphi) : (E, \varphi) \in W_0^+,$
$\varphi(V) \subseteq V,\ \dim V = k\}$

Theo (Hausel–Hitchin 2022) W_k^+ closed and $h|_{W_k^+}$ proper but no longer one to one.

Example: For (E, φ_0) there is only one possible V.

$$\rightsquigarrow E' = \mathcal{O}(-c) \oplus K(-c) \oplus \dots \oplus K^{-n+k} \oplus \dots \oplus K^{-n+1}$$
$$\oplus K^{-n+k+1}(-c) \oplus$$

$$\varphi_0' = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots s_c & \ddots & \\ & & \ddots 1 & 0 \end{pmatrix}$$

In general we consider

$$\mathcal{H}_K := \{((E, \varphi_a), V) : \varphi_a|_c(V) \subseteq V\} \subseteq W_0^+ \times Gr(k, n)$$

Then (by definition)

$$\begin{tikzcd} \mathcal{H}_K \arrow[r, "\cong"] \arrow[d, "\pi_1"] & W_K^+ \arrow[d, "h|_{W_K^+}"] \\ W_0^+ \arrow[r, "h|_{W_0^+}"] & \mathcal{A} \end{tikzcd}$$

$$Gl_n/P_K \qquad P_K = Stab(V_0) = \{(\ast|\ast)\}$$

On the other hand $\{(X, V) \in S \times \overbrace{Gr(k, n)}^{Gl_n/P_K} : X(V) \subseteq V\}$

$$\begin{tikzcd} \mathcal{H}_K \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & S_K \arrow[r, "(\ast)"] \arrow[d, "p_1"] \arrow[rr, bend right, "\sim", "(\ast\ast\ast)"'] & \mathfrak{p}_K/\!/P_K \arrow[r, phantom, "\overset{(\ast\ast)}{\cong}"] & \mathfrak{t}_K/\!/L_K \arrow[d] \\ W_0^+ \arrow[r, "res|_c"] & S \arrow[rr, "\sim"] & & \mathfrak{g}/\!/G \end{tikzcd}$$

(*): $x \in \mathfrak{p}_v = \mathrm{Stab}(v)$ so we can map to $\mathfrak{p}_v /\!/ P_v \cong \mathfrak{l}_v /\!/ L_v$.

Then $\mathfrak{l}_v /\!/ L_v \cong \mathfrak{l}_K /\!/ L_K$ canonically.

(**): X a Richardson element $([\mathfrak{p}, X] = \mathfrak{n})$ then
$\rho(\mathfrak{n}) = \rho(X) \rightsquigarrow \rho$ is constant on $\mathfrak{n}$.

(***) top map is finite because $\downarrow$ finite $\xrightarrow{\sim}$ $\downarrow$ finite.
Moreover the degrees of those two to the sides agree $\rightsquigarrow$ has degree 1 $\Rightarrow$ isom.

Finally

$$\mathbb{C}[\mathfrak{l}_K]^{L_K} \cong H_{L_K}(pt) \cong H_{P_K}(pt) \cong$$

$$\cong H(BP_K) \cong H\left(EG/P_K\right) \cong H\left(\frac{EG \times (G/P_K)}{G}\right) \cong$$

$$gx \longleftarrow\!\shortmid (x, gP)$$

$$\cong H_G(G/P_K).$$

So $\mathfrak{h}|_{w_K^+}$ is modelled on $\mathrm{Spec}\, H_{GL_n}(Gr(k,n))$

$$\downarrow$$

$\mathrm{Spec}\, H_{GL_n}(\cdot)$.