Cyclic Higgs bundles and the Toledo invariant

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G is a complex semisimple group, $\mathfrak{g}$ its Lie algebra.

- Study the subvarieties of **cyclic Higgs bundles** inside the moduli space $\mathcal{M}(G)$ of polystable G-Higgs bundles on smooth projective curve C.

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- Recall that a G-Higgs bundle (E, φ) is a holomorphic principal G-bundle over C with a section $\varphi \in H^0(C, E(\mathfrak{g}) \otimes K_C)$.
- E.g $G = \mathrm{SL}_n$ then E is a rank n vector bundle with $\det E = \mathcal{O}_C$ and $\varphi : E \to E \otimes K_C$ a traceless bundle morphism.

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Definition

Let $\theta \in \mathrm{Aut}_m(G)$ be an order m automorphism, $\zeta \in \mathbb{C}^\times$ a primitive m-th root of unity. **Cyclic Higgs bundles** are the fixed points of the $\mathbb{Z}/m\mathbb{Z}$-action generated by

$$(E, \varphi) \mapsto (\theta(E), \zeta^k d\theta(\varphi)).$$

Vinberg θ-pairs

- θ induces a $\mathbb{Z}/m\mathbb{Z}$-grading

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- The adjoint representation restricts to a representation of any closed subgroup H with $G_0 \subseteq H \subseteq G_\theta = N_G(G^\theta)$, e.g. $H = G^\theta$, on $\hat{\mathfrak{g}}_k$.

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- $(H, \hat{\mathfrak{g}}_k)$ is called **Vinberg θ-pair**.
- (E, φ) with E an H-bundle and $\varphi \in H^0(E(\hat{\mathfrak{g}}_k) \otimes K_C)$ is called $(H, \hat{\mathfrak{g}}_k)$-**Higgs pair**. Moduli spaces $\mathcal{M}(H, \hat{\mathfrak{g}}_k)$.

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Theorem (García-Prada–Ramanan, 2019)

The image of $\mathcal{M}(G^\theta, \hat{\mathfrak{g}}_k) \to \mathcal{M}(G)$ consists of cyclic Higgs bundles. All stable and simple cyclic Higgs bundles for θ and k are obtained by using the θ', up to equivalence, on the same outer class as θ.

More motivation

- We will focus on the study of $\mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$.
- G_0 because it is easier to work with. If G is s.c. then $G_0 = G^\theta$.
- $\hat{\mathfrak{g}}_1$ because if we had $\hat{\mathfrak{g}}_k$ for any other $k \neq 0$ we can pass to a subalgebra, and for $k = 0$ it is just $\mathcal{M}(G_0)$.

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- For $m = 2$ we can find an antiholomorphic involution τ for the compact form, with $\tau\theta = \theta\tau =: \sigma$. Let $G^{\mathbb{R}} := G^\sigma$ be the real form. Then $\mathcal{M}(G_0, \hat{\mathfrak{g}}_1) = \mathcal{M}(G^{\mathbb{R}})$, $G^{\mathbb{R}}$-Higgs bundles.

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- For even $m = 2m'$ they live inside the Lagrangians fixed by $(\theta^{m'}(E), -d\theta^{m'}(\varphi))$.
- Related to different constructions such as certain local systems (Simpson, 2006), solutions to the *affine Toda equations* (Baraglia, 2015), *cyclic surfaces* (Labourie, 2017)...

Vinberg $\mathbb{C}^\times$-pairs

- We will **make use of the theory of $\mathbb{Z}$-gradings** of the Lie algebra $\mathfrak{g}$.

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

- Similarly we have representations $(G_0, \mathfrak{g}_k)$ called Vinberg $\mathbb{C}^\times$-pairs. We will use $(G_0, \mathfrak{g}_1)$.
- The corresponding Higgs bundles inside $\mathcal{M}(G)$ are the **Hodge bundles**, i.e. fixed points of the $\mathbb{C}^\times$-action (Simpson, 1992).

Relating the gradings

- How do we get $\mathbb{Z}$-gradings to appear in our setting?
- From a $\mathbb{Z}$-grading we can project the indices to $\mathbb{Z}/m\mathbb{Z}$ and get a $\mathbb{Z}/m\mathbb{Z}$-grading. (i.e. from $\mathbb{C}^\times \to \mathrm{Aut}(\mathfrak{g})$ we precompose with $\mu_m \to \mathbb{C}^\times$).

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- We want $\hat{\mathfrak{g}}_0 = \mathfrak{g}_0$ (in order to have the same structure group G_0) so we will look at gradings

$$\mathfrak{g} = \mathfrak{g}_{1-m} \oplus \cdots \oplus \mathfrak{g}_{m-1}.$$

- Then $\hat{\mathfrak{g}}_j = \mathfrak{g}_j \oplus \mathfrak{g}_{j-m}$ for $j \neq 0$.

Examples

- **Real forms of Hermitian type.** Let $G^{\mathbb{R}} \subseteq G$ be a real form of Hermitian type. $H^{\mathbb{R}} \subseteq G^{\mathbb{R}}$ its maximal compact subgroup.
- This means that $G^{\mathbb{R}}/H^{\mathbb{R}}$ is a Hermitian symmetric space so we get a decomposition of the complexified tangent space at the identity, $\hat{\mathfrak{g}}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ in the $\pm i$-eigenspaces of the complex structure.

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- This results in $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ inducing the $\mathbb{Z}/2\mathbb{Z}$-grading, as desired.
- They are: $SU(p,q)$, $SO(2,n)$, $Sp(2n,\mathbb{R})$, $SO^*(2n)$, $E_6(-14)$ and $E_7(-25)$.

Examples II

- **Quaternion-Kähler symmetric spaces.** Let $G^{\mathbb{R}} \subseteq G$ be a real form of *quaternionic* type. $H^{\mathbb{R}} \subseteq G^{\mathbb{R}}$ its maximal compact subgroup.
- By this we mean that $G^{\mathbb{R}}/H^{\mathbb{R}}$ is a quaternion-Kähler symmetric space, i.e. its holonomy is contained in $\mathrm{Sp}(n)\,\mathrm{Sp}(1) \subseteq \mathrm{SO}(4n)$.

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- In this case we get decompositions of the symmetric pair $\hat{\mathfrak{g}}_0 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\hat{\mathfrak{g}}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ given by $\mathrm{ad}(I)$ where $I \in \hat{\mathfrak{g}}_0$ is one of the almost complex structures.
- We can then consider the associated cyclic grading for $m = 3$. Note that for this grading $\mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$ is **not** $\mathcal{M}(G^{\mathbb{R}})$.

Examples III

- The quaternionic $\mathbb{Z}$-grading from before exists in every type.
- Alternate construction: fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ and simple roots $\Pi \subseteq \Delta(\mathfrak{g}, \mathfrak{t}) =: \Delta$. Consider the highest root $\beta \in \Delta$.
- Normalise the dual Killing form so that $B^*(\beta, \beta) = 2$. Then for any other root we have $B^*(\alpha, \beta) \in \{-2, -1, 0, 1, 2\}$.

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- This induces the quaternionic $\mathbb{Z}$-grading by assigning degree $B^*(\alpha, \beta)$ to $\mathfrak{g}_\alpha$.
- The corresponding real forms are $SU(2, n)$, $SO(4, n)$, $\mathrm{Sp}(2, 2n)$, $E_6(2)$, $E_7(-5)$, $E_8(-24)$, $F_4(4)$ and $G_2(2)$.

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$$\mathfrak{g}_k = \mathfrak{sl}_n \cap \bigoplus_j \mathrm{End}(V_j, V_{j+k}).$$

- $G_0 = S(\mathrm{GL}_{d_0} \times \cdots \times \mathrm{GL}_{d_{m-1}})$, and $\mathfrak{g}_1$ endomorphisms of the form:

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- A_m **quiver (or *linear* quiver) representations**.
- For $m = 2$, the Hermitian form $SU(d_0, d_1)$. For $m = 3$ and dimensions $(1, n, 1)$ it gives the quaternionic grading in type A.

Examples V

- The associated $\mathbb{Z}/m\mathbb{Z}$-grading is:

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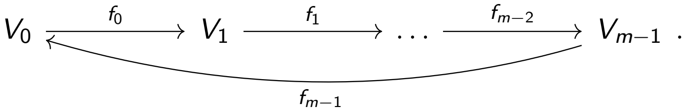

- **Cyclic quiver representations**.

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- Example: linear quiver grading for dimensions $(2, 1, 1)$. If the simple roots are $\alpha_i = e_{i+1} - e_i$ then we have root vectors $E_{\alpha_i} = E_{i+1,i}$ so the labelling is

$$\bullet_0 \;\text{———}\; \bullet_1 \;\text{———}\; \bullet_1$$

(In general we will have a 1 each d_i dots).

Classification of $\mathbb{Z}/m\mathbb{Z}$-gradings

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- First for inner $\theta \in \mathrm{Int}_m \mathfrak{g}$. Let $\alpha_0 := -\beta$ be the lowest root and consider the Dynkin diagram for $\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$, i.e. the **affine Dynkin diagram**.

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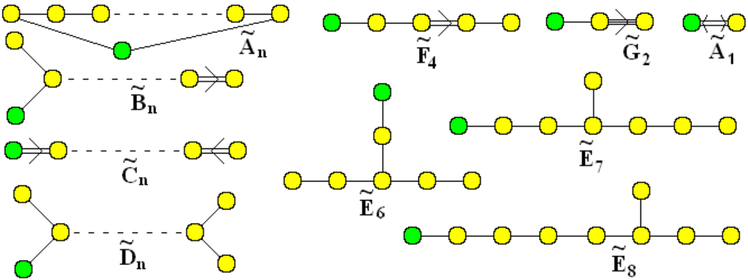

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- A labelling $\{p_i\}$ of the affine Dynkin diagram corresponds to a $\mathbb{Z}/m\mathbb{Z}$-grading where $m = \sum_i n_i p_i$ and the n_i are the smallest such that $0 = \sum_i n_i \alpha_i$.

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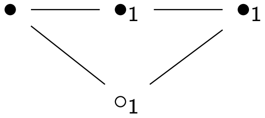

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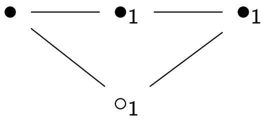

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- In both cases we can now clearly see that it comes from a $\mathbb{Z}$-grading given by looking at the regular Dynkin diagram inside of the affine one.

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Proposition

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In particular, if the lowest root can be carried to a simple root with $p_i > 0$ via an automorphism of the affine Dynkin diagram ($\Longleftrightarrow$ we have $n_i = 1$ and $p_i > 0$ for some i), we can always find such a grading. E.g. we can always do it in type A.

Classification of $\mathbb{Z}/m\mathbb{Z}$-gradings IV

- What about gradings that are not inner?
- They are classified by labellings of **Kac diagrams**, which are obtained by taking a Dynkin automorphism s in the outer class and considering the action of the disconnected torus $S = T \times \mathbb{Z}/q\mathbb{Z}$ where $T \subseteq C_{\mathrm{Aut}(\mathfrak{g})}(s)$ is a maximal torus and $\mathbb{Z}/q\mathbb{Z}$ acts via s (so $q = \mathrm{ord}(s)$). An affine Dynkin diagram is constructed for this action to give the Kac diagram.

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- These never come from $\mathbb{Z}$-gradings because the map $\mathbb{C}^\times \to \mathrm{Aut}(\mathfrak{g})$ giving a $\mathbb{Z}$-grading goes into $\mathrm{Int}(\mathfrak{g})$.

The Toledo character

- Recall the setup: G complex semisimple, $\theta \in \mathrm{Aut}_m(G)$ inducing a $\mathbb{Z}/m\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$ coming from a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{1-m} \oplus \cdots \oplus \mathfrak{g}_{m-1}$ with grading element D. Let B be the Killing form or any Ad-invariant non-degenerate bilinear form.

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Definition

The **Toledo character** $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by $x \mapsto \lambda_B \cdot B(D, x)$.

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- Recall the setup: G complex semisimple, $\theta \in \mathrm{Aut}_m(G)$ inducing a $\mathbb{Z}/m\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$ coming from a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{1-m} \oplus \cdots \oplus \mathfrak{g}_{m-1}$ with grading element D. Let B be the Killing form or any Ad-invariant non-degenerate bilinear form.

Definition

The **Toledo character** $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by $x \mapsto \lambda_B \cdot B(D, x)$.

- Here $\lambda_B \in \mathbb{C}^\times$ is a constant that makes it independent of the choice of B. If we want this to generalise the Toledo character for Hermitian real forms (Biquard–García-Prada–Rubio, 2017) we can choose $\lambda_B := B^*(\gamma, \gamma)$ where γ is the longest root labelled with a 1 in the $\mathbb{Z}$-grading.

The Toledo rank

- The first thing we can define with χ_T is a rank associated with every G_0-orbit in $\mathfrak{g}_1$.

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$$\mathrm{rk}_T(e) := \frac{\chi_T(h)}{2},$$

where (h, e, f) is an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$.

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where (h, e, f) is an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$.

- It is well defined and constant on G_0 orbits.
- There are finitely many orbits so it is bounded. The maximum is denoted by $\mathrm{rk}_T(G_0, \mathfrak{g}_1)$ and is attained precisely at the unique open orbit $\Omega \subseteq \mathfrak{g}_1$. The minimum is 0.

The Toledo rank II

- For example, in the linear quivers $\mathbb{Z}$-grading, an element $e \in \mathfrak{g}_1$ is defined by maps $f_i : V_i \to V_{i+1}$. The Toledo rank is a linear combinantion of the ranks of $f_{r,s} := f_s \circ \cdots \circ f_{r+1} \circ f_r$. If $m = 2$ it is just $\operatorname{rk} f$ where $f \colon V_0 \to V_1$.

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- Since it only depends on the G_0 orbit, we can define the Toledo rank of a Higgs field $\varphi \in H^0(E(\mathfrak{g}_1) \otimes K_C)$ appearing in a $(G_0, \mathfrak{g}_1)$-Higgs bundle by:

$$\operatorname{rk}_T(\varphi) := \operatorname{rk}_T(\varphi(c)),$$

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  for generic $c \in C$.
- In our case of interest, $\mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$, we can decompose the Higgs field $\varphi = \varphi^+ + \varphi^-$ according to the decomposition $\hat{\mathfrak{g}}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_{1-m}$ and compute $\operatorname{rk}_T(\varphi^+)$.

The Toledo invariant

- Choose a positive integer $q \in \mathbb{Z}_{>0}$ so that $q\chi_T$ lifts to $\tilde{\chi}_T : G_0 \to \mathbb{C}^\times$.

Definition

Let (E, φ) be a $(G_0, \hat{\mathfrak{g}}_1)$-Higgs pair. Its **Toledo invariant** is defined as

$$\tau(E, \varphi) := \frac{\deg(E \times_{\tilde{\chi}_T} \mathbb{C}^\times)}{q}.$$

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- It was introduced and studied in the spaces $\mathcal{M}(G_0, \mathfrak{g}_1)$ in (Biquard–Collier–García-Prada–Toledo, 2023).
- Generalises the Toledo invariant for Higgs bundles for Hermitian real forms studied both from the representation and Higgs bundle points of view in (Turaev, 1984), (Domic–Toledo, 1987), (Bradlow–García-Prada–Gothen, 2001 & 2003), (Burger–Iozzi–Wienhard 2003 & 2010), (Biquard–García-Prada–Rubio, 2017).

The Toledo invariant, example

- Consider the grading for cyclic quiver representations with dimensions d_i. In this case a $\mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$-Higgs bundle can be seen as a rank n vector bundle $E := E_0 \oplus \cdots \oplus E_{m-1}$ with rank $E_i = d_i$ and $\det E = \mathcal{O}_C$, and a bundle morphism $\varphi : E \to E \otimes K_C$ such that $\varphi(E_j) \subseteq E_{j+1} \otimes K_C$ (with cyclic indices).

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- In this case

$$\tau(E, \varphi) = 2 \sum_{j=0}^{m-1} (j - \alpha) \deg E_j,$$

where $\alpha = \frac{\sum_j d_j}{d_j}$.

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- For $m = 2$ one gets $\tau = 2\frac{d_0 \deg E_1 - d_1 \deg E_0}{d_0 + d_1}$. Using that $\deg E_0 = - \deg E_1$ it becomes $\tau = 2 \deg E_1$, the Toledo invariant for $SU(d_0, d_1)$-Higgs bundles.

Milnor-type inequality

Theorem (García-Prada–G., 2023)

Let (E,φ) *be a* (λD)*-semistable* $(G_0,\hat{\mathfrak{g}}_1)$*-Higgs pair,* $\lambda\in\mathbb{R}$*. Then*

$$\tau(E,\varphi)\geqslant -\operatorname{rk}_T(\varphi^+)(2g-2)+\lambda(B^*(\gamma,\gamma)B(D,D)-\operatorname{rk}_T(\varphi^+)).$$

- In particular, if $(E,\varphi)\in\mathcal{M}(G_0,\hat{\mathfrak{g}}_1)$, then $\tau(E,\varphi)\geqslant -(2g-2)\operatorname{rk}_T(\varphi^+)\geqslant -(2g-2)\operatorname{rk}_T(G_0,\mathfrak{g}_1)$.
- This contains previous Milnor-type inequalities (Hermitian real forms, $(G_0,\mathfrak{g}_1)$-Higgs pairs) and extends to the more general situation that we are considering.

Milnor-type inequality

Theorem (García-Prada–G., 2023)

Let (E, φ) be a (λD)-semistable $(G_0, \hat{\mathfrak{g}}_1)$-Higgs pair, $\lambda \in \mathbb{R}$. Then

$$\tau(E, \varphi) \geqslant -\operatorname{rk}_T(\varphi^+)(2g-2) + \lambda(B^*(\gamma, \gamma)B(D, D) - \operatorname{rk}_T(\varphi^+)).$$

- In particular, if $(E, \varphi) \in \mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$, then $\tau(E, \varphi) \geqslant -(2g-2)\operatorname{rk}_T(\varphi^+) \geqslant -(2g-2)\operatorname{rk}_T(G_0, \mathfrak{g}_1)$.
- This contains previous Milnor-type inequalities (Hermitian real forms, $(G_0, \mathfrak{g}_1)$-Higgs pairs) and extends to the more general situation that we are considering.
- Proof uses the existence of a relative invariant for the Toledo character, i.e. a rational map $F: \mathfrak{g}_1 \to \mathbb{C}$ such that $F(g \cdot v) = \tilde{\chi}_T(g)F(v)$.
- For the quiver representations case one can prove it *by hand* but it is very tedious.

Extremal Toledo invariant

- For which $(G_0, \hat{\mathfrak{g}}_1)$-Higgs pairs is the bound $\tau(E, \varphi) = -(2g-2)\,\mathrm{rk}_T(G_0, \mathfrak{g}_1)$ attained?

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- This locus inside $\mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$ is denoted by $\mathcal{M}_{max}(G_0, \hat{\mathfrak{g}}_1)$.
- For $\mathcal{M}(G^{\mathbb{R}})$ where $G^{\mathbb{R}}$ is Hermitian of **tube type** the answer is given by the **Cayley correspondence**:

Theorem (Biquard–García-Prada–Rubio, 2017)

If $G^{\mathbb{R}}$ is a Hermitian real form of tube type, there exists $G^* \subseteq G^\theta$ (the noncompact dual) such that if the order of $\exp(2\pi i D) \in G_0$ divides $(2g-2)$, then:

$$\mathcal{M}_{max}(G^{\mathbb{R}}) \simeq \mathcal{M}_{K_C^2}(G^*).$$

Cayley correspondence for $U(n, n)$

- For example, consider $G^{\mathbb{R}} = U(n, n) \subseteq \mathrm{GL}_{2n}$, which is Hermitian of tube type.
- The Cayley partner is $G^{*} = \mathrm{GL}_n$.
- The corresponding $\mathbb{Z}/2\mathbb{Z}$-grading is the cyclic quiver one in GL_{2n} for dimensions (n, n).

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- A $U(n,n)$-Higgs bundle $(E,\varphi) \in \mathcal{M}(U(n,n))$ is a vector bundle $E = E_0 \oplus E_1$ with rank $E_i = n$ and a bundle map $\varphi : E \to E \otimes K_C$ with $\varphi(E_j) \subseteq E_{j+1} \otimes K_C$.
- Let $\varphi_j : E_j \to E_{j+1} \otimes K_C$. Then the Toledo invariant is extremal if φ_0 is an isomorphism. Then $E_0 \simeq E_1 \otimes K_C$.

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- Let $\varphi_j : E_j \to E_{j+1} \otimes K_C$. Then the Toledo invariant is extremal if φ_0 is an isomorphism. Then $E_0 \simeq E_1 \otimes K_C$.
- The Cayley correspondence is given by $(E,\varphi) \mapsto (E_1, \varphi_0\varphi_1)$.

Studying $\mathcal{M}_{max}(G_0, \hat{\mathfrak{g}}_1)$

- From now on, we will restrict to **JM-regular** $(G_0, \mathfrak{g}_1)$. This condition generalises the tube type as well as the other conditions in the Cayley correspondence for Hermitian real forms.

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- For example, the quiver grading when the vector (d_j) is palindromic and unimodal.
- Recall the unique open orbit $\Omega \subseteq \mathfrak{g}_1$, characterised by $\mathrm{rk}_T(\Omega) = \mathrm{rk}_T(G_0, \mathfrak{g}_1)$.
- $\tau(E, \varphi)$ attains the bound if and only if $\varphi^+(c) \in \Omega$ for all $c \in C$.
- Are there any such Higgs pairs?

Canonical uniformising Higgs pair

- Let $T = \mathbb{C}^\times$ and denote by E_T the frame bundle of $K_C^{\frac{-1}{2}}$.
- The **canonical uniformising** SL_2-**Higgs bundle** is $(E_T(\mathrm{SL}_2), e)$, where (h, e, f) spans $\mathfrak{sl}_2$ and we identify $T = \exp(h) \subseteq \mathrm{SL}_2$. Note that $E_T(\langle e \rangle) \otimes K_C \simeq \mathcal{O}_C$ so e is a Higgs field.

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- In terms of vector bundles, it is $K_C^{\frac{1}{2}} \oplus K_C^{\frac{-1}{2}}$ with the Higgs field $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
- Now pick $e \in \Omega \subseteq \mathfrak{g}_1$. By Jacobson–Morozov we can choose an $\mathfrak{sl}_2$-triple (h, e, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$. This yields $\mathrm{SL}_2 \hookrightarrow G$ (or $\mathrm{PSL}_2 \hookrightarrow G$).
- The resulting bundle $(E_T(G_0), e)$ lives in $\mathcal{M}_{max}(G_0, \hat{\mathfrak{g}}_1)$ by construction.

Cayley partner

- Let $e \in \Omega \subseteq \mathfrak{g}_1$ and $Z := C_{G_0}(e)$.
- Let $V := \mathrm{Im}(\mathrm{ad}(e)^{m-1}|_{\mathfrak{g}_{1-m}} : \mathfrak{g}_{1-m} \to \mathfrak{g}_0)$. The map $\psi := \mathrm{ad}(e)^{m-1}|_{\mathfrak{g}_{1-m}}$ is an isomorphism onto V.

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- Equivalently, V is $\mathfrak{g}_0 \cap W$ where W are the $(2m-1)$-dimensional irreducible representations of $\mathfrak{sl}_2$ in $\mathfrak{g}$ given by (h, e, f).
- The adjoint representation of $Z \subseteq G_0$ on $\mathfrak{g}_0$ has V as a subrepresentation.
- We will consider K_C^m-twisted (Z, V)-Higgs pairs $\mathcal{M}_{K_C^m}(Z, V)$.

Cayley map

- We define the **Cayley map** by starting from a K_C^m-twisted (Z, V)-Higgs pair (E_Z, φ') and sending it to:

$$E := (E_T \otimes E_Z)(G_0)$$

$$\varphi := e + \psi^{-1}(\varphi').$$

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  (We use that T and Z are commuting subgroups of G_0).
- As a map from isomorphism classes of K_C^m-twisted (Z, V)-Higgs pairs to $(G_0, \hat{\mathfrak{g}}_1)$-Higgs pairs with Toledo invariant equal to $-(2g-2)\,\mathrm{rk}_T(G_0, \mathfrak{g}_1)$ it is **bijective**.
- Remains to see if it **preserves polystability.**

Cayley correspondence

- The inverse of the map given before (i.e. going from $(G_0, \hat{\mathfrak{g}}_1)$ to (Z, V)) can be directly seen to preserve stability.
- The other direction is harder, in fact requires the gauge theoretical point of view (Hitchin–Kobayashi correspondence) and some assumption on (Z, V).

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- The other direction is harder, in fact requires the gauge theoretical point of view (Hitchin–Kobayashi correspondence) and some assumption on (Z, V).

Theorem (García-Prada–G., 2023)

The Cayley map restricts to an embedding

$$\mathcal{M}_{max}(G_0, \hat{\mathfrak{g}}_1) \to \mathcal{M}_{K_C^m}(Z, V).$$

If (Z, V) is a Vinberg θ-pair, the previous embedding is an isomorphism.

- Generalises the Cayley correspondence for Hermitian real forms of tube type (Biquard–García-Prada–Rubio, 2017) and for $(G_0, \mathfrak{g}_1)$–Higgs pairs (Biquard–Collier–García-Prada–Toledo, 2023).

Cayley correspondence example

- Consider the quiver grading of dimensions $(1,1,1)$ in GL_3.
- Fixing $V_0 \simeq V_1$ and $V_1 \simeq V_2$ gives $e \in \Omega \subseteq \mathfrak{g}_1$. Thus $Z = C_{G_0}(e) = \mathbb{C}^\times$ (embedded diagonally). V is a one-dimensional space corresponding to the weight 0 representation of $\mathbb{C}^\times$. Also (Z, V) is a Vinberg pair.

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- The space of Higgs pairs with extremal Toledo invariant is $\mathcal{M}_{K_C^3}(Z, V)$ which consists of pairs (L, φ) where L is a line bundle over C and $\varphi \in H^0(K_C^3)$. I.e. it is $\mathrm{Pic}(C) \times H^0(K_C^3)$.

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- Such a pair (L, φ) corresponds to $E = L \otimes (K_C \oplus \mathcal{O}_C \oplus K_C^{-1})$ with Higgs field $\begin{pmatrix} 0 & 0 & \varphi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Application to the quaternionic grading

- Consider the pair $(G_0, \hat{\mathfrak{g}}_1)$ for the quaternionic / highest root grading.
- In this case we can use that $\dim \mathfrak{g}_{-2} = 1$ and that $(G_0, \mathfrak{g}_{-2})$ is JM-regular to obtain:

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Theorem (García-Prada–G., 2023)

Let $(E, \varphi) \in \mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$. Then

$$-(8g-8) \leqslant \tau(E,\varphi) \leqslant 4g-4.$$

The bounds are attained except in type C, where we have

$$-(2g-2) \leqslant \tau(E,\varphi) \leqslant 2g-2.$$

Application to the quaternionic grading II

- Similarly, except for type C, the pair $(G_0, \mathfrak{g}_1)$ is always JM-regular and (Z, V) is always a Vinberg pair (since $\dim V = 1$). Thus:

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Theorem (García-Prada–G., 2023)

Let $(E, \varphi) \in \mathcal{M}(G_0, \hat{\mathfrak{g}}_1)$ and assume that we are not in type C. Then

$$\mathcal{M}_{max}(G_0, \hat{\mathfrak{g}}_1) \simeq \mathcal{M}_{K_C^3}(Z, V).$$

- The example of the quiver grading of dimensions $(1, 1, 1)$ from before is one of them.

Thank you!!

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