Very stable regular G-Higgs bundles

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G-Higgs bundles

G connected semisimple complex Lie group, Lie algebra $\mathfrak{g}$

X compact Riemann surface, $g \geqslant 2$, canonical K_X

A G-**Higgs bundle** (E, φ) over X:

- E a principal G-bundle over X
- φ a holomorphic section of $E(\mathfrak{g}) \otimes K_X$ (**Higgs field**)

Moduli space of G-Higgs bundles

- **Moduli space** of polystable G-Higgs bundles $\mathcal{M}(G)$.
- Hyperkähler metric $\rightsquigarrow$ symplectic strucutre ω (on smooth locus)
- **Hitchin map** $h_G : \mathcal{M}(G) \to \mathcal{A}(G) := \bigoplus_i H^0(C, K_X^{d_i})$, proper.
- Natural $\mathbb{C}^\times$-**action**:

$$(E, \varphi) \mapsto (E, \lambda\varphi)$$

- Limits when $\lambda \to 0$ exist and are fixed.

Definition

The **upward flow** of the fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$:

$$W^+_{(E,\varphi)} := \left\{ (E', \varphi') : \lim_{\lambda \to 0} (E', \lambda\varphi') = (E, \varphi) \right\} \subseteq \mathcal{M}(G)$$

Complex Lagrangian subvarieties (BAA-brane, mirror in $\mathcal{M}(G^\vee)$)

Very stable G-Higgs bundles

Definition (Hausel–Hitchin, 2022)

Smooth fixed point $(E, \varphi) \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ is **very stable** if $W^+_{(E,\varphi)} \cap h_G^{-1}(0) = \{(E, \varphi)\}$. Equivalently, if $W^+_{(E,\varphi)} \subseteq \mathcal{M}(G)$ is closed. Otherwise, it is **wobbly**.

- Drinfeld and Laumon: E stable G-bundle $\rightsquigarrow (E, 0)$ very stable $\iff$ no nonzero nilpotent $\varphi \in H^0(E(\mathfrak{g}) \otimes K_X)$.
- Motivation: Identifies simplest $\mathbb{C}^\times$-invariant closed complex Lagrangians. h_G is proper on $W^+_{(E,\varphi)}$, easier study of mirror.

Question

Can we describe which fixed points are very stable?

Example $G = \mathrm{PGL}_n(\mathbb{C})$

Take $G = \mathrm{PGL}_n(\mathbb{C})$ (or $\mathrm{GL}_n(\mathbb{C})$). E is a rank n vector bundle, $\varphi : E \to E \otimes K_X$ traceless.

Fixed points are related to **chains**: $E = \bigoplus_{j=1}^m E_j$, $\varphi(E_j) \subseteq E_{j+1} \otimes K_X$.

$$E_1 \xrightarrow{\varphi_1} E_2 K_X \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{m-1}} E_m K_X^{m-1}$$

The **type** of a fixed point is $(\mathrm{rk}\, E_1, \dots, \mathrm{rk}\, E_m)$.

Example $G = \mathrm{PGL}_n(\mathbb{C})$ II

Type (n): $(E, 0)$ with E stable vector bundle. Nonempty open dense subset of very stable (Laumon, 1988).

Type $(1, 1, \ldots, 1)$: (Hausel–Hitchin, 2022) Simplest example:

$$E = \mathcal{O} \oplus K_X^{-1} \oplus \cdots \oplus K_X^{-n+1}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \qquad \varphi_a = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_n \\ 1 & 0 & \ldots & 0 & -a_{n-1} \\ 0 & 1 & \ldots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

for $a = (a_i)_{i=2}^n \in \mathcal{A}(G)$. (E, φ_0) is fixed, upward flow is $\{(E, \varphi_a)\}_a$
Hitchin section $\rightsquigarrow$ very stable.

Example $G = \mathrm{PGL}_n(\mathbb{C})$ III

In general, fixed points of type $(1, 1, \dots, 1)$ are

$$E = \mathcal{O} \oplus K_X^{-1}(D_1) \oplus K_X^{-2}(D_1 + D_2) \oplus \cdots \oplus K_X^{-n+1}(D_1 + \cdots + D_{n-1})$$

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varphi_1 & 0 & \dots & 0 & 0 \\ 0 & \varphi_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_{n-1} & 0 \end{pmatrix}$$

with $\mathrm{div}(\varphi_j) = D_j$.

Theorem (Hausel–Hitchin, 2022)

Stable fixed point (E, φ) of type $(1, 1, \dots, 1)$ is very stable if and only if $D_1 + \cdots + D_{n-1}$ is reduced.

Example $G = \mathrm{PGL}_n(\mathbb{C})$ IV and $G = \mathrm{SO}_{2n}(\mathbb{C})$

- Other types: (Peón–Nieto, 2024), e.g. type $(n_1, n_2) \notin \{(1,1),(2,1),(1,2)\}$ are wobbly.
- Other groups? $G = \mathrm{SO}_{2n}(\mathbb{C})$. One *type* is

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$$\varphi = \left(\begin{array}{cccc|cccc}
0 & & & & & & & \\
\varphi_1 & \ddots & & & & & & \\
 & \ddots & 0 & & & & & \\
 & & \varphi_{n-1} & 0 & & & & \\
\hline
 & & \varphi_n & 0 & 0 & & & \\
 & & & -\varphi_n^t & -\varphi_{n-1}^t & \ddots & & \\
 & & & & & \ddots & 0 & \\
 & & & & & & -\varphi_1^t & 0
\end{array}\right).$$

Fixed points in $\mathcal{M}(G)$

Fixed points given by $\mathbb{Z}$-gradings:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

with $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$.
$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0 \rightsquigarrow G_0 \subseteq G$ connected subgroup.
$[\mathfrak{g}_0, \mathfrak{g}_j] \subseteq \mathfrak{g}_j \rightsquigarrow$ **representation** $G_0 \to \mathrm{GL}(\mathfrak{g}_j)$.

Prop. (Simpson 1988, Biquard–Collier–García-Prada–Toledo, 2023)

Fixed points $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ characterised by:

- E reduces to E_{G_0} a G_0-bundle.
- $\varphi \in H^0(E_{G_0}(\mathfrak{g}_j) \otimes K_X)$ for $j \neq 0$

Called $(G_0, \mathfrak{g}_j)$-**Higgs pairs.**

Example of $\mathbb{Z}$-grading

$G = \mathrm{PGL}_n(\mathbb{C})$. Gradings of $\mathfrak{sl}_n(\mathbb{C})$ given by *dividing matrices in blocks* with squares in the diagonal.

$$\left(\begin{array}{c|c|c|c} \mathfrak{g}_0 & \mathfrak{g}_{-1} & \cdots & \mathfrak{g}_{1-m} \\ \hline \mathfrak{g}_1 & \mathfrak{g}_0 & \cdots & \mathfrak{g}_{2-m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathfrak{g}_{m-1} & \mathfrak{g}_{m-2} & \cdots & \mathfrak{g}_0 \end{array}\right).$$

I.e. by block size choices $(n_1, \ldots, n_m)$.

$$G_0 = \mathrm{P}(\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_m}(\mathbb{C})) = \mathrm{P}(\mathrm{Aut}(V_1) \times \cdots \times \mathrm{Aut}(V_m))$$

$$\mathfrak{g}_1 = \bigoplus_{i=1}^{m-1} \mathrm{Hom}(V_i, V_{i+1}).$$

A $(G_0, \mathfrak{g}_1)$-**Higgs pair** precisely defines a **chain** of type $(n_1, \ldots, n_m)$.

Fixed points of Borel type

Question (recall)

Can we describe which fixed points are very stable?

Focus on **regular nilpotent** Higgs field. These are of **Borel type**.

Definition

Let $T \subseteq G$ be a maximal torus and $\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ be a system of simple roots. The **Borel grading** is defined by $\mathfrak{t} = \mathfrak{g}_0$ and $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{g}_1$.

Definition

A fixed point (E, φ) is of **Borel type** if it reduces to $(G_0, \mathfrak{g}_1)$ for the Borel grading.

Borel type for $G = \mathrm{PGL}_n(\mathbb{C})$

Take $G = \mathrm{PGL}_n(\mathbb{C})$.

$$\left(\begin{array}{c|c|c|c} \mathfrak{g}_0 & \mathfrak{g}_{-1} & \cdots & \mathfrak{g}_{1-m} \\ \hline \mathfrak{g}_1 & \mathfrak{g}_0 & \cdots & \mathfrak{g}_{2-m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathfrak{g}_{m-1} & \mathfrak{g}_{m-2} & \cdots & \mathfrak{g}_0 \end{array}\right).$$

Borel grading (G_0 a torus) has block sizes $(1, 1, \ldots, 1)$. Recall:

$$E = \mathcal{O} \oplus K_X^{-1}(D_1) \oplus K_X^{-2}(D_1 + D_2) \oplus \cdots \oplus K_X^{-n+1}(D_1 + \cdots + D_{n-1})$$

$$\varphi = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \varphi_1 & 0 & \ldots & 0 & 0 \\ 0 & \varphi_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \varphi_{n-1} & 0 \end{pmatrix}$$

with $\mathrm{div}(\varphi_j) = D_j$. Very stable $\iff$ $D_1 + \cdots + D_{n-1}$ reduced (Hausel–Hitchin, 2022).

Borel type for $G = SO_{2n}(\mathbb{C})$

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$$\varphi = \left(\begin{array}{cccc|cccc}
0 & & & & & & & \\
\varphi_1 & \ddots & & & & & & \\
 & \ddots & 0 & & & & & \\
 & & \varphi_{n-1} & 0 & & & & \\
\hline
 & & \varphi_n & 0 & 0 & & & \\
 & & & -\varphi_n^t & -\varphi_{n-1}^t & \ddots & & \\
 & & & & & \ddots & 0 & \\
 & & & & & & -\varphi_1^t & 0
\end{array}\right).$$

We get divisors $D_j := \mathrm{div}(\varphi_j)$.

Proposition

Very stable $\iff$ $D_1 + 2D_2 + \cdots + 2D_{n-2} + D_{n-1} + D_n$ is reduced.

Borel type for $G = SO_{2n+1}(\mathbb{C})$

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus \mathcal{O}_C \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$$\varphi = \begin{pmatrix} 0 & & & & & & & & \\ \varphi_1 & \ddots & & & & & & & \\ & \ddots & 0 & & & & & & \\ & & \varphi_{n-1} & 0 & & & & & \\ & & & \varphi_n & 0 & & & & \\ & & & & -\varphi_n^t & 0 & & & \\ & & & & & -\varphi_{n-1}^t & \ddots & & \\ & & & & & & \ddots & 0 & \\ & & & & & & & -\varphi_1^t & 0 \end{pmatrix}.$$

We get divisors $D_j := \mathrm{div}(\varphi_j)$.

Proposition

Fixed point is very stable $\iff$ $D_1 + 2D_2 + \cdots + 2D_n$ is reduced.

Borel type for $G = Sp_{2n}(\mathbb{C})$

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*)$$

$$\varphi = \left(\begin{array}{cccc|cccc} 0 & & & & & & & \\ \varphi_1 & \ddots & & & & & & \\ & \ddots & 0 & & & & & \\ & & \varphi_{n-1} & 0 & & & & \\ \hline & & & \varphi_n & 0 & & & \\ & & & & -\varphi_{n-1}^t & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & -\varphi_1^t & 0 \end{array}\right).$$

We get divisors $D_j := \mathrm{div}(\varphi_j)$.

Proposition

Fixed point is very stable $\iff$ $2D_1 + \cdots + 2D_{n-1} + D_n$ is reduced.

Multiplicity divisor

Why? General phenomenon.

- $\mathfrak{g}_1 = \bigoplus_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}$
- $G_0 = T \curvearrowright \mathfrak{g}_{\alpha_i}$
- Map $E(\mathfrak{g}_1) \to E(\mathfrak{g}_{\alpha_i})$ gives $\varphi_i \in H^0(E(\mathfrak{g}_{\alpha_i}) \otimes K_X)$ sections of line bundles.
- Divisors $D_i := \mathrm{div}(\varphi_i)$.

Fundamental coweights: $\{\omega_1^\vee, \dots, \omega_r^\vee\} \subseteq \mathfrak{t}$, dual basis to $\Pi \subseteq \mathfrak{t}^*$.

Definition

For a fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ of Borel type, we define its **multiplicity divisor:**

$$\mu(E, \varphi) := \sum_{i=1}^{r} D_i \omega_i^\vee .$$

Divisor on X whose coefficients are (dominant) coweights.

Minuscule coweights

- There is a **partial ordering** on the space of dominant coweights of $\mathfrak{g}$.

$$\lambda \geqslant \mu \iff \lambda - \mu \text{ is a sum of positive coroots .}$$

- Minimal dominant coweights are called **minuscule**. Representations of $\mathfrak{g}^\vee$ with a single Weyl orbit of weights.

For example:

- $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ has the trivial (0) and every k-th exterior power of the standard ($\omega_k^\vee$).
- $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ has the trivial (0), the standard ($\omega_1^\vee$), and two more ($\omega_{n-1}^\vee, \omega_n^\vee$).
- E_8, F_4, G_2 only have the trivial (0).

Classification theorem

Recall:

- $(E, \varphi) \in \mathcal{M}^{s\mathbb{C}^\times}(G)$ smooth fixed point of Borel type.
- $\varphi \rightsquigarrow \varphi_i \rightsquigarrow D_i \rightsquigarrow \mu(E, \varphi)$ a dominant coweight at each point.
- Notion of minimality for dominant coweights.

Theorem

Let $(E, \varphi) \in \mathcal{M}^{s\mathbb{C}^\times}(G)$ be a smooth fixed point of Borel type. It is very stable if and only if $\mu(E, \varphi)|_x$ is minuscule at every $x \in X$.

Consequences

- Concrete descriptions in classical groups from before.
- We can say which components (of Borel type) have very stable points. Indeed, the **topological type** of the fixed point in $\pi_1(G_0) \simeq \mathbb{Z}^r$ is determined by $\deg \mu(E, \varphi) := \sum_{x \in X} \mu(E, \varphi)|_x$.
- (Hausel–Hitchin, 2022) Every component has very stable points for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Indeed all $\omega_i^\vee$ are minuscule in this case.
- Not at all for other $\mathfrak{g}$. Extreme case: E_8, F_4, G_2 require regularity everywhere.

Thank you!