Very stable regular G-Higgs bundles

Miguel González (ICMAT)

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Stratifications of Higgs bundle moduli spaces and related topics
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G-Higgs bundles

G connected semisimple complex group, Lie algebra $\mathfrak{g}$

C smooth projective complex curve, $g \geqslant 2$, canonical K_C

A G-**Higgs bundle** (E, φ) over C:

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C$ (**Higgs field**)

Moduli space of G-Higgs bundles

- **Moduli space** of polystable G-Higgs bundles $\mathcal{M}(G)$.
- Hyperkähler metric $\rightsquigarrow$ symplectic strucutre ω (on smooth locus)
- **Hitchin map** $h_G : \mathcal{M}(G) \to \mathcal{A}(G) := \bigoplus_i H^0(C, K_C^{d_i})$, proper.
- Natural $\mathbb{C}^\times$-**action**:

$$(E, \varphi) \mapsto (E, \lambda\varphi)$$

- **Semiprojective:** proper fixed point locus, limits at 0 exist.

Definition

The **upward flow** of the fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$:

$$W^+_{(E,\varphi)} := \left\{ (E', \varphi') : \lim_{\lambda \to 0} (E', \lambda\varphi') = (E, \varphi) \right\} \subseteq \mathcal{M}(G)$$

Complex Lagrangian subvarieties (BAA-brane, mirror in $\mathcal{M}(G^\vee)$)

Very stable G-Higgs bundles

Definition (Hausel–Hitchin, 2022)

Smooth fixed point $(E, \varphi) \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ is **very stable** if $W^+_{(E,\varphi)} \cap h_G^{-1}(0) = \{(E, \varphi)\}$. Equivalently, if $W^+_{(E,\varphi)} \subseteq \mathcal{M}(G)$ is closed. Otherwise, it is **wobbly**.

- Drinfeld and Laumon: E stable G-bundle $\rightsquigarrow$ $(E, 0)$ very stable $\iff$ no nonzero nilpotent $\varphi \in H^0(E(\mathfrak{g}) \otimes K_C)$.
- Motivation: Identifies simplest $\mathbb{C}^\times$-invariant closed complex Lagrangians. h_G is proper on $W^+_{(E,\varphi)}$, easier study of mirror.

Question

Can we describe which fixed points are very stable?

Example $G = \mathrm{PGL}_n(\mathbb{C})$

Take $G = \mathrm{PGL}_n(\mathbb{C})$ (or $\mathrm{GL}_n(\mathbb{C})$). E is a rank n vector bundle, $\varphi : E \to E \otimes K_C$ traceless.

Fixed points are related to **chains**: $E = \bigoplus_{j=1}^m E_j$, $\varphi(E_j) \subseteq E_{j+1} \otimes K_C$.

$$E_1 \xrightarrow{\varphi_1} E_2 K_C \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{m-1}} E_m K_C^{m-1}$$

The **type** of a fixed point is $(\mathrm{rk}\, E_1, \dots, \mathrm{rk}\, E_m)$.

Example $G = \mathrm{PGL}_n(\mathbb{C})$ II

Type (n): $(E, 0)$ with E stable vector bundle. Nonempty open dense subset of very stable (Laumon, 1988).

Type $(1, 1, \dots, 1)$: Hausel–Hitchin, 2022. Simplest example:

$$E = \mathcal{O} \oplus K_C^{-1} \oplus \cdots \oplus K_C^{-n+1}$$

$$\varphi_0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \qquad \varphi_a = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & \dots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

for $a = (a_i)_{i=2}^n \in \mathcal{A}(G)$. (E, φ_0) is fixed, upward flow is $\{(E, \varphi_a)\}_a$ **Hitchin section** $\rightsquigarrow$ very stable.

Example $G = \mathrm{PGL}_n(\mathbb{C})$ III

In general, fixed points of type $(1, 1, \ldots, 1)$ are

$$E = \mathcal{O} \oplus K_C^{-1}(D_1) \oplus K_C^{-2}(D_1 + D_2) \oplus \cdots \oplus K_C^{-n+1}(D_1 + \cdots + D_{n-1})$$

$$\varphi = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \varphi_1 & 0 & \ldots & 0 & 0 \\ 0 & \varphi_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \varphi_{n-1} & 0 \end{pmatrix}$$

with $\mathrm{div}(\varphi_j) = D_j$.

Theorem (Hausel–Hitchin, 2022)

Stable fixed point (E, φ) of type $(1, 1, \ldots, 1)$ is very stable if and only if $D_1 + \cdots + D_{n-1}$ is reduced.

Example $G = \mathrm{PGL}_n(\mathbb{C})$ IV and $G = \mathrm{SO}_{2n}(\mathbb{C})$

- Other types: (Peón–Nieto, 2024), e.g. type $(n_1, n_2) \notin \{(1,1),(2,1),(1,2)\}$ are wobbly.
- Other groups? $G = \mathrm{SO}_{2n}(\mathbb{C})$. One *type* is

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$$\varphi = \left(\begin{array}{cccc|cccc}
0 & & & & & & & \\
\varphi_1 & \ddots & & & & & & \\
 & \ddots & 0 & & & & & \\
 & & \varphi_{n-1} & 0 & & & & \\
\hline
 & & \varphi_n & 0 & 0 & & & \\
 & & & -\varphi_n^t & -\varphi_{n-1}^t & \ddots & & \\
 & & & & & \ddots & 0 & \\
 & & & & & & -\varphi_1^t & 0
\end{array}\right).$$

Fixed points in $\mathcal{M}(G)$

Fixed points given by $\mathbb{Z}$-gradings:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

with $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$.
$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0 \rightsquigarrow G_0 \subseteq G$ connected subgroup.
$[\mathfrak{g}_0, \mathfrak{g}_j] \subseteq \mathfrak{g}_j \rightsquigarrow$ **representation** $G_0 \to \mathrm{GL}(\mathfrak{g}_j)$.

Prop. (Simpson 1988, Biquard–Collier–García-Prada–Toledo, 2023)

Fixed points $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ characterised by:

- E reduces to E_{G_0} a G_0-bundle.
- $\varphi \in H^0(E_{G_0}(\mathfrak{g}_j) \otimes K_C)$ for $j \neq 0$

Called $(G_0, \mathfrak{g}_j)$-**Higgs pairs.**

Example of $\mathbb{Z}$-grading

$G = \mathrm{PGL}_n(\mathbb{C})$. Gradings of $\mathfrak{sl}_n(\mathbb{C})$ given by *dividing matrices in blocks* with squares in the diagonal.

$$\left(\begin{array}{c|c|c|c} \mathfrak{g}_0 & \mathfrak{g}_{-1} & \cdots & \mathfrak{g}_{1-m} \\ \hline \mathfrak{g}_1 & \mathfrak{g}_0 & \cdots & \mathfrak{g}_{2-m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathfrak{g}_{m-1} & \mathfrak{g}_{m-2} & \cdots & \mathfrak{g}_0 \end{array}\right).$$

I.e. by block size choices $(n_1, \dots, n_m)$.

$$G_0 = \mathrm{P}(\mathrm{GL}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_m}(\mathbb{C})) = \mathrm{P}(\mathrm{Aut}(V_1) \times \cdots \times \mathrm{Aut}(V_m))$$

$$\mathfrak{g}_1 = \bigoplus_{i=1}^{m-1} \mathrm{Hom}(V_i, V_{i+1}).$$

A $(G_0, \mathfrak{g}_1)$-**Higgs pair** precisely defines a **chain** of type $(n_1, \dots, n_m)$.

Fixed points of Borel type

Question (recall)

Can we describe which fixed points are very stable?

Focus on **regular nilpotent** Higgs field. These are of **Borel type**.

Definition

Let $T \subseteq G$ be a maximal torus and
$\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ be a system of simple roots. The **Borel grading** is defined by $\mathfrak{t} = \mathfrak{g}_0$ and $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{g}_1$.

Definition

A fixed point (E, φ) is of **Borel type** if it reduces to $(G_0, \mathfrak{g}_1)$ for the Borel grading.

Borel type for $G = \mathrm{PGL}_n(\mathbb{C})$

Take $G = \mathrm{PGL}_n(\mathbb{C})$.

$$\left(\begin{array}{c|c|c|c} \mathfrak{g}_0 & \mathfrak{g}_{-1} & \cdots & \mathfrak{g}_{1-m} \\ \hline \mathfrak{g}_1 & \mathfrak{g}_0 & \cdots & \mathfrak{g}_{2-m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathfrak{g}_{m-1} & \mathfrak{g}_{m-2} & \cdots & \mathfrak{g}_0 \end{array}\right).$$

Borel grading (G_0 a torus) has block sizes $(1, 1, \dots, 1)$. Recall:

$E = \mathcal{O} \oplus K_C^{-1}(D_1) \oplus K_C^{-2}(D_1 + D_2) \oplus \cdots \oplus K_C^{-n+1}(D_1 + \cdots + D_{n-1})$

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varphi_1 & 0 & \dots & 0 & 0 \\ 0 & \varphi_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_{n-1} & 0 \end{pmatrix}$$

with $\mathrm{div}(\varphi_j) = D_j$. Very stable $\iff$ $D_1 + \cdots + D_{n-1}$ reduced (Hausel–Hitchin, 2022).

Borel type for $G = SO_{2n}(\mathbb{C})$

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$$\varphi = \left(\begin{array}{cccc|cccc}
0 & & & & & & & \\
\varphi_1 & \ddots & & & & & & \\
 & \ddots & 0 & & & & & \\
 & & \varphi_{n-1} & 0 & & & & \\
\hline
 & & \varphi_n & 0 & 0 & & & \\
 & & & -\varphi_n^t & -\varphi_{n-1}^t & \ddots & & \\
 & & & & & \ddots & 0 & \\
 & & & & & & -\varphi_1^t & 0
\end{array}\right).$$

We get divisors $D_j := \mathrm{div}(\varphi_j)$.

Proposition

Very stable $\iff$ $D_1 + 2D_2 + \cdots + 2D_{n-2} + D_{n-1} + D_n$ is reduced.

Borel type for $G = SO_{2n+1}(\mathbb{C})$

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus \mathcal{O}_C \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$$\varphi = \begin{pmatrix} 0 & & & & & & & & \\ \varphi_1 & \ddots & & & & & & & \\ & \ddots & 0 & & & & & & \\ & & \varphi_{n-1} & 0 & & & & & \\ & & & \varphi_n & 0 & & & & \\ & & & & -\varphi_n^t & 0 & & & \\ & & & & & -\varphi_{n-1}^t & \ddots & & \\ & & & & & & \ddots & 0 & \\ & & & & & & & -\varphi_1^t & 0 \end{pmatrix}.$$

We get divisors $D_j := \mathrm{div}(\varphi_j)$.

Proposition

Fixed point is very stable $\iff$ $D_1 + 2D_2 + \cdots + 2D_n$ is reduced.

Borel type for $G = Sp_{2n}(\mathbb{C})$

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*)$$

$$\varphi = \left(\begin{array}{cccc|cccc} 0 & & & & & & & \\ \varphi_1 & \ddots & & & & & & \\ & \ddots & 0 & & & & & \\ & & \varphi_{n-1} & 0 & & & & \\ \hline & & & \varphi_n & 0 & & & \\ & & & & -\varphi_{n-1}^t & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & -\varphi_1^t & 0 \end{array}\right).$$

We get divisors $D_j := \mathrm{div}(\varphi_j)$.

Proposition

Fixed point is very stable $\iff$ $2D_1 + \cdots + 2D_{n-1} + D_n$ is reduced.

Multiplicity divisor

Why? General phenomenon.

- $\mathfrak{g}_1 = \bigoplus_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}$
- $G_0 = T \curvearrowright \mathfrak{g}_{\alpha_i}$
- Map $E(\mathfrak{g}_1) \to E(\mathfrak{g}_{\alpha_i})$ gives $\varphi_i \in H^0(E(\mathfrak{g}_{\alpha_i}) \otimes K_C)$ sections of line bundles.
- Divisors $D_i := \mathrm{div}(\varphi_i)$.

Fundamental coweights: $\{\omega_1^\vee, \ldots, \omega_r^\vee\} \subseteq \mathfrak{t}$, dual basis to $\Pi \subseteq \mathfrak{t}^*$.

Definition

For a fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ of Borel type, we define its **multiplicity divisor:**

$$\mu(E, \varphi) := \sum_{i=1}^{r} D_i \omega_i^\vee.$$

Divisor on C whose coefficients are (dominant) coweights.

Minuscule coweights

- There is a **partial ordering** on the space of dominant coweights of $\mathfrak{g}$.

$$\lambda \geqslant \mu \iff \lambda - \mu \text{ is a sum of positive coroots} .$$

- Minimal dominant coweights are called **minuscule**. Representations of $\mathfrak{g}^\vee$ with a single Weyl orbit of weights.

For example:

- $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ has the trivial (0) and every k-th exterior power of the standard $(\omega_k^\vee)$.
- $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ has the trivial (0), the standard $(\omega_1^\vee)$, and two more $(\omega_{n-1}^\vee, \omega_n^\vee)$.
- E_8, F_4, G_2 only have the trivial (0).

Classification theorem

Recall:

- $(E, \varphi) \in \mathcal{M}^{s\mathbb{C}^\times}(G)$ smooth fixed point of Borel type.
- $\varphi \rightsquigarrow \varphi_i \rightsquigarrow D_i \rightsquigarrow \mu(E, \varphi)$ a dominant coweight at each point.
- Notion of minimality for dominant coweights.

Theorem

Let $(E, \varphi) \in \mathcal{M}^{s\mathbb{C}^\times}(G)$ be a smooth fixed point of Borel type. It is very stable if and only if $\mu(E, \varphi)|_c$ is minuscule at every $c \in C$.

Consequences

- Concrete descriptions in classical groups from before.
- We can say which components (of Borel type) have very stable points. Indeed, the **topological type** of the fixed point in $\pi_1(G_0) \simeq \mathbb{Z}^r$ is determined by $\deg \mu(E, \varphi) := \sum_{c \in C} \mu(E, \varphi)|_c$.
- (Hausel–Hitchin, 2022) Every component has very stable points for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Indeed all $\omega_i^\vee$ are minuscule in this case.
- Not at all for other $\mathfrak{g}$. Extreme case: E_8, F_4, G_2 require regularity everywhere.

Strategy: Hecke transformations

Fixed points of Borel type can be related by **Hecke transformations**. Fix $c \in C$ and let $C_0 := C \setminus \{c\}$.

Definition

A **Hecke transformation** of a G-Higgs bundle (E, φ) at c is (E', φ', ψ) where (E', φ') is another G-Higgs bundle together with an isomorphism

$$\psi : (E', \varphi')|_{C_0} \xrightarrow{\sim} (E, \varphi)|_{C_0}.$$

- Trivialise E over C_0 as well as over a formal disk $C_1 \simeq \operatorname{Spec} \mathbb{C}[[z]]$ around $c \in C$.
- Transition function over $C_{01} := C_0 \cap C_1 \simeq \operatorname{Spec} \mathbb{C}((z))$.

$$f_E : C_{01} \to G.$$

We fix this data.

The affine Grassmannian

- Hecke transformations: change the transition function $f_E \in G((z))$ by $f_E\sigma$ for $\sigma \in G((z))$.
- If $\sigma \in G[[z]]$, result is isomorphic. Therefore

$$\left\{\begin{matrix}\text{Hecke transformations}\\ \text{of (E,0) at c}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{points in } \textbf{affine Grassmannian}\\ \mathrm{Gr}_G := G((z))/G[[z]]\end{matrix}\right\}$$

(non canonically). See (Wong, 2013).

Affine Springer fibre

- What about φ?
- Locally, $\varphi_1 : C_1 \to \mathfrak{g}$, i.e. in $\mathfrak{g}[[z]]$.
- Given $\sigma \in G((z))$, it transforms to $\mathrm{Ad}_{\sigma^{-1}} \varphi_1 \in \mathfrak{g}((z))$, a priori only over C_{01}.
- We then ask

$$\mathrm{Ad}_{\sigma^{-1}} \varphi_1 \in \mathfrak{g}[[z]],$$

which defines the **affine Springer fibre** over φ_1.

$$\left\{ \begin{array}{c} \text{Hecke transformations} \\ \text{of } (E, \varphi) \text{ at } c \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{points in an affine Springer fibre} \\ \text{given by } \varphi \text{ in } \mathrm{Gr}_G\,. \end{array} \right\}.$$

$\mathbb{C}^\times$-action on Gr_G

- $\mathbb{C}^\times$-action on the affine Springer fibre such that if σ gives (E', φ') then $\lambda \cdot \sigma$ gives $(E', \lambda\varphi')$.
- Its fixed points (cocharacters of $T = G_0$) produce $\mathbb{C}^\times$-fixed Higgs bundles.
- Can produce curves between $\mathbb{C}^\times$-fixed Higgs bundles from curves in the affine Springer fibre.

Wobbly fixed points

- Consider $(E, \varphi) \in \mathcal{M}^{s\mathbb{C}^\times}(G)$ smooth fixed point of Borel type with $\mu(E, \varphi)|_c = \mu$ not minuscule.
- There is a positive coroot $\alpha^\vee \in \Delta_+^\vee$ with $\mu - \alpha^\vee$ dominant.
- In the affine Grassmannian, there is an (explicit) curve connecting the identity and $\alpha^\vee$.
- Hecke transformation produces curve connecting (E, φ) with (E', φ') such that $\mu(E', \varphi')|_c = \mu - \alpha^\vee$.
- Can choose $\alpha^\vee$ so that the curve is in the affine Springer fibre and stability is preserved.

Very stable fixed points

- Now $\mu(E,\varphi)|_c$ is minuscule for all $c \in C$.
- Wobbly means there is a $\mathbb{C}^\times$-invariant curve connecting (E,φ) with another fixed point.
- This curve **comes from Hecke transformation** of a similar curve flowing to $(E',\varphi') \in \mathcal{M}(G)^{s\mathbb{C}^\times}$, a Hecke transformation of (E,φ) such that $\mu(E',\varphi')$ has smaller support.
- Arrive at the case $\mu(E,\varphi) = 0$ (everywhere regular φ) which is very stable.
- The **key step** (third point) only works because in the minuscule cases the relevant part of the affine Springer fibre is a single point.

Virtual equivariant multiplicities

Invariant of fixed point components defined and studied in Hausel–Hitchin, 2022.

Definition

The **virtual equivariant multiplicity** of a smooth fixed point $(E, \varphi) \in \mathcal{M}^{s\mathbb{C}^\times}(G)$ is

$$m(E, \varphi) := \frac{\chi(\mathrm{Sym}((W^+_{(E,\varphi)})^*))}{\chi(\mathrm{Sym}(\mathcal{A}(G)^*))} \in \mathbb{Z}((t)).$$

- It is constant along the fixed point components.
- If (E, φ) is very stable, it is a monic, palindromic polynomial with non-negative coefficients and $m(E, \varphi)|_{t=1} \in \mathbb{Z}_{\geqslant 1}$ is the multiplicity of the component in the nilpotent cone $h_G^{-1}(0)$.

Virtual equivariant multiplicities of Borel type

We can use our description in terms of gradings to obtain the $m(E, \varphi)$ for Borel type in terms of $\dim \mathfrak{g}_j$ and $\mu(E, \varphi)$. Here are the very stable ones with μ supported at a point.

Type of $\mathfrak{g}$	μ	$m_{\mathcal{E}}(t)$
A_n	$\omega_i^\vee$	$\prod_{j=1}^{i} \frac{1-t^{n-j+1}}{1-t^j}$
B_n	$\omega_1^\vee$	$1 + t + \cdots + t^{2n-1}$
C_n	$\omega_n^\vee$	$\prod_{j=1}^{n}(1 + t^j)$
D_n	$\omega_1^\vee$	$(1 + t^{n-1})(1 + t + \cdots + t^{n-1})$
	$\omega_{n-1}^\vee, \omega_n^\vee$	$\prod_{j=1}^{n}(1 + t^j)$
E_6	$\omega_1^\vee, \omega_6^\vee$	$(1 + t^4 + t^8)(1 + t + \cdots + t^8)$
E_7	$\omega_7^\vee$	$(1 + t^5)(1 + t^9)(1 + t + \cdots + t^{13})$

These computations appeared in (Hausel–Hitchin, 2022), leading to conjecture that minuscule representations were related to very stable Higgs bundles.

Virtual equivariant multiplicities and Dynkin polynomials

- The virtual equivariant multiplicities shown above agree with the **Dynkin polynomials**

$$\mathcal{D}_\mu(t) := \prod_{\alpha \in \Delta^+} \frac{1 - t^{\alpha(\rho^\vee + \mu)}}{1 - t^{\alpha(\rho^\vee)}}.$$

- Recall that components of Borel type may not have very stable points. The corresponding virtual equivariant multiplicities (experimentally) are not polynomials. This is in contrast to other types (e.g. (n_1, n_2) for $\mathrm{GL}_n(\mathbb{C})$, Peón–Nieto 2024).

Thank you!