Very stable G-Higgs bundles with regular nilpotent Higgs field

Miguel González (ICMAT)
(with Oscar García-Prada and Tamás Hausel)

Reunión de la Red Temática de Geometría y Física

Noviembre 2024

INSTITUTO DE CIENCIAS MATEMÁTICAS

"la Caixa" Foundation

G-Higgs bundles

G complex semisimple Lie group ($\mathrm{SL}_n\,\mathbb{C}, \mathrm{PGL}_n\mathbb{C}, \mathrm{Sp}_{2n}\,\mathbb{C}, \mathrm{SO}_n\,\mathbb{C}, \mathrm{G}_2, \dots$), Lie algebra $\mathfrak{g}$

C smooth projective complex curve, $g \geqslant 2$, canonical K_C

A G-**Higgs bundle** (E, φ) over C:

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C$ (**Higgs field**)

Examples

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C$

Examples:

- $G = \mathrm{SL}_n \mathbb{C}$. Then $V := E(\mathbb{C}^n)$ a **vector bundle of rank** n. $\det V = \mathcal{O}_C$. Since $\mathfrak{g} = \mathrm{End}_0(\mathbb{C}^n)$, we get $\varphi : V \to V \otimes K_C$ traceless.
- $G = \mathrm{SO}_n \mathbb{C}$. $V := E(\mathbb{C}^n)$ a **vector bundle of rank** n with a **non-degenerate symmetric form** $Q : V \xrightarrow{\sim} V^*$. This time $\varphi : V \to V \otimes K_C$ with $\varphi^t = -\varphi$.

Very stable G-Higgs bundles

Moduli space $\mathcal{M}(G)$ of isomorphism classes of polystable G-Higgs bundles.

Natural $\mathbb{C}^\times$-action:

$$(E, \varphi) \mapsto (E, \lambda\varphi)$$

The **upward flow** of the fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$:

$$W^+_{(E,\varphi)} := \left\{ (E', \varphi') : \lim_{\lambda \to 0} (E', \lambda\varphi') = (E, \varphi) \right\} \subseteq \mathcal{M}(G)$$

Complex lagrangian subvariety (BAA-brane, mirror in $\mathcal{M}(G^\vee)$)

Definition (Hausel–Hitchin, 2022)

Stable fixed point (E, φ) is **very stable** if $W^+_{(E,\varphi)} \subseteq \mathcal{M}(G)$ is closed

Fixed points

Fixed points of **Borel type**:
(Regular nilpotent Higgs field)
We will need:

- Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$
- Simple roots $\{\alpha_1, \ldots, \alpha_r\} \subseteq \Delta \rightsquigarrow \mathfrak{g}_{\alpha_i}$.
- $\mathfrak{g}_1 := \bigoplus \mathfrak{g}_{\alpha_i}$

We take (E, φ) such that

- E reduces to a T-bundle.
- φ section of $E(\mathfrak{g}_1) \otimes K_C$.

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$\mathfrak{g} = \mathfrak{sl}_4\mathbb{C}$$

$$\begin{pmatrix} a & b & c & d & 0 \\ f & g & h & 0 & -d \\ i & j & 0 & -h & -c \\ k & 0 & -j & -g & -b \\ 0 & -k & -i & -f & -a \end{pmatrix}$$

$$\mathfrak{g} = \mathfrak{so}_5\mathbb{C}$$

Examples

- $G = \mathrm{SL}_4\,\mathbb{C}$
  V vector bundle, rank 4 $\rightsquigarrow$ $V = L_0 \oplus L_1 \oplus L_2 \oplus L_3$
  $\varphi : V \to V \otimes K_C \rightsquigarrow \varphi(L_j) \subseteq L_{j+1} \otimes K_C$

$$L_0 \xrightarrow{\varphi_1} L_1 K_C \xrightarrow{\varphi_2} L_2 K_C^2 \xrightarrow{\varphi_3} L_3 K_C^3$$

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$\mathfrak{g} = \mathfrak{sl}_4\mathbb{C}$$

- $G = \mathrm{SO}_5\,\mathbb{C}$
  V orthogonal bundle, rank 5 $\rightsquigarrow$ $V = L_0 \oplus L_1 \oplus \mathcal{O}_C \oplus L_1^* \oplus L_0^*$
  $\varphi : V \to V \otimes K_C$, $\varphi^t = -\varphi \rightsquigarrow$

Multiplicities

$\mathfrak{g}_1 = \bigoplus \mathfrak{g}_{\alpha_i}$, projection:

$$\pi_i : E(\mathfrak{g}_1) \to E(\mathfrak{g}_{\alpha_i})$$

Higgs field gives **components** φ_i, sections of each $E(\mathfrak{g}_{\alpha_i}) \otimes K_C$

Definition

The **multiplicity coweight** at $c \in C$ is

$$\mu_c := \sum_{i=1}^{r} (\mathrm{ord}_c \varphi_i) \cdot \omega_i^\vee$$

where $\omega_1^\vee, \ldots, \omega_r^\vee \in \mathfrak{t}$ are the the fundamental coweights.

Classification

Multiplicity coweight $\mu_c := \sum_{i=1}^r (\mathrm{ord}_c \varphi_i) \cdot \omega_i^\vee$

(Dominant) coweight μ is **minuscule** if $\alpha(\mu) \in \{0, \pm 1\}$ for all roots $\alpha \in \Delta$.

Theorem

Stable fixed point (E, φ) of Borel type is very stable

$$\iff$$

Multiplicity coweights μ_c are all minuscule

Main technique is the action of (subspaces of) the **affine Grassmannian** Gr_G on G-Higgs bundles by **Hecke transformations**.

Examples and consequences

- $G = \mathrm{SL}_4\,\mathbb{C} \rightsquigarrow V = L_0 \oplus L_1 \oplus L_2 \oplus L_3$

$$L_0 \xrightarrow{\varphi_1} L_1 K_C \xrightarrow{\varphi_2} L_2 K_C^2 \xrightarrow{\varphi_3} L_3 K_C^3$$

Minuscule coweights: $0, \omega_1^\vee, \omega_2^\vee, \omega_3^\vee$
Very stable $\iff$ $\varphi_1, \varphi_2, \varphi_3$ no common zeroes, and simple (Hausel–Hitchin, 2022)

- $G = \mathrm{SO}_5\,\mathbb{C} \rightsquigarrow V = L_0 \oplus L_1 \oplus \mathcal{O}_C \oplus L_1^* \oplus L_0^*$.

$$L_0 \xrightarrow{\varphi_1} L_1 K_C \xrightarrow{\varphi_2} \mathcal{O}_C K_C^2 \xrightarrow{-\varphi_2^*} L_1^* K_C^3 \xrightarrow{-\varphi_1^*} L_0^* K_C^4$$

Minuscule coweights: $0, \omega_1^\vee$
Very stable $\iff$ φ_2 has no zeroes and φ_1 simple zeroes

- $G = G_2, F_4, E_8$. Minuscule coweight: 0
Very stable: **"Hitchin section"**

Thank you!