Very stable *G*-Higgs bundles with regular nilpotent Higgs field

Miguel González (ICMAT) (with Oscar García-Prada and Tamás Hausel)

Reunión de la Red Temática de Geometría y Física

Noviembre 2024





G-Higgs bundles

G complex semisimple Lie group $(SL_n\mathbb{C}, PGL_n\mathbb{C}, Sp_{2n}\mathbb{C}, SO_n\mathbb{C}, G_2, ...)$, Lie algebra \mathfrak{g}

C smooth projective complex curve, $g \geqslant 2$, canonical K_C

A *G*-**Higgs bundle** (E, φ) over *C*:

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_{\mathcal{C}}$ (Higgs field)

Examples

- E a principal G-bundle over C
- φ a section of $E(\mathfrak{g}) \otimes K_C$

Examples:

- $G = \operatorname{SL}_n \mathbb{C}$. Then $V := E(\mathbb{C}^n)$ a vector bundle of rank n. det $V = \mathcal{O}_C$. Since $\mathfrak{g} = \operatorname{End}_0(\mathbb{C}^n)$, we get $\varphi : V \to V \otimes K_C$ traceless.
- $G = SO_n \mathbb{C}$. $V := E(\mathbb{C}^n)$ a vector bundle of rank n with a non-degenerate symmetric form $Q : V \xrightarrow{\sim} V^*$. This time $\varphi : V \to V \otimes K_C$ with $\varphi^t = -\varphi$.

Very stable G-Higgs bundles

Moduli space $\mathcal{M}(G)$ of isomorphism classes of polystable *G*-Higgs bundles.

Natural \mathbb{C}^{\times} -action:

$$(E,\varphi)\mapsto (E,\lambda\varphi)$$

The **upward flow** of the fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^{\times}}$:

$$W_{(E,\varphi)}^+ := \left\{ (E',\varphi') : \lim_{\lambda \to 0} (E',\lambda \varphi') = (E,\varphi) \right\} \subseteq \mathcal{M}(G)$$

Complex lagrangian subvariety (BAA-brane, mirror in $\mathcal{M}(\mathcal{G}^{\vee})$)

Definition (Hausel-Hitchin, 2022)

Stable fixed point (E,φ) is **very stable** if $W_{(E,\varphi)}^+\subseteq \mathcal{M}(G)$ is closed

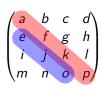
Fixed points

Fixed points of **Borel type**: (Regular nilpotent Higgs field) We will need:

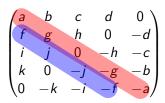
- ullet Cartan subalgebra ${\mathfrak t}\subseteq {\mathfrak g}$
- Simple roots $\{\alpha_1, \ldots, \alpha_r\} \subseteq \Delta \leadsto \mathfrak{g}_{\alpha_i}$.
- $\mathfrak{g}_1 := \bigoplus \mathfrak{g}_{\alpha_i}$

We take (E, φ) such that

- E reduces to a T-bundle.
- φ section of $E(\mathfrak{g}_1) \otimes K_C$.



$$\mathfrak{g}=\mathfrak{sl}_4\mathbb{C}$$

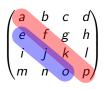


$$\mathfrak{g}=\mathfrak{so}_5\mathbb{C}$$

Examples

• $G = \operatorname{SL}_4 \mathbb{C}$ V vector bundle, rank $4 \rightsquigarrow V = L_0 \oplus L_1 \oplus L_2 \oplus L_3$ $\varphi : V \to V \otimes K_C \rightsquigarrow \varphi(L_j) \subseteq L_{j+1} \otimes K_C$

$$L_0 \xrightarrow{\varphi_1} L_1 \mathcal{K}_C \xrightarrow{\varphi_2} L_2 \mathcal{K}_C^2 \xrightarrow{\varphi_3} L_3 \mathcal{K}_C^3$$



$$\mathfrak{g}=\mathfrak{sl}_4\mathbb{C}$$

• $G = SO_5 \mathbb{C}$ V orthogonal bundle, rank $5 \rightsquigarrow V = L_0 \oplus L_1 \oplus \mathcal{O}_C \oplus L_1^* \oplus L_0^*$ $\varphi : V \to V \otimes K_C$, $\varphi^t = -\varphi \rightsquigarrow$

Multiplicities

 $\mathfrak{g}_1 = \bigoplus \mathfrak{g}_{\alpha_i}$, projection:

$$\pi_i: E(\mathfrak{g}_1) \to E(\mathfrak{g}_{\alpha_i})$$

Higgs field gives **components** φ_i , sections of each $E(\mathfrak{g}_{\alpha_i}) \otimes K_C$

Definition

The multiplicity coweight at $c \in C$ is

$$\mu_c := \sum_{i=1}^r (\operatorname{ord}_c \varphi_i) \cdot \omega_i^{\vee}$$

where $\omega_1^{\vee}, \dots, \omega_r^{\vee} \in \mathfrak{t}$ are the fundamental coweights.

Classification

Multiplicity coweight $\mu_c := \sum_{i=1}^r (\operatorname{ord}_c \varphi_i) \cdot \omega_i^{\vee}$

(Dominant) coweight μ is **minuscule** if $\alpha(\mu) \in \{0, \pm 1\}$ for all roots $\alpha \in \Delta$.

Theorem

Stable fixed point (E, φ) of Borel type is very stable



Multiplicity coweights μ_c are all minuscule

Main technique is the action of (subspaces of) the **affine Grassmannian** Gr_G on G-Higgs bundles by **Hecke** transformations.

Examples and consequences

• $G = SL_4 \mathbb{C} \rightsquigarrow V = L_0 \oplus L_1 \oplus L_2 \oplus L_3$

$$L_0 \xrightarrow{\varphi_1} L_1 K_C \xrightarrow{\varphi_2} L_2 K_C^2 \xrightarrow{\varphi_3} L_3 K_C^3$$

Minuscule coweights: $0, \omega_1^{\vee}, \omega_2^{\vee}, \omega_3^{\vee}$ Very stable $\iff \varphi_1, \varphi_2, \varphi_3$ no common zeroes, and simple (Hausel–Hitchin, 2022)

• $G = SO_5 \mathbb{C} \leadsto V = L_0 \oplus L_1 \oplus \mathcal{O}_C \oplus L_1^* \oplus L_0^*$.

$$L_0 \xrightarrow{\varphi_1} L_1 K_C \xrightarrow{\varphi_2} \mathcal{O}_C K_C^2 \xrightarrow{-\varphi_2^*} L_1^* K_C^3 \xrightarrow{-\varphi_1^*} L_0^* K_C^4$$

Minuscule coweights: $0, \omega_1^\vee$ Very stable $\iff \varphi_2$ has no zeroes and φ_1 simple zeroes

• $G = G_2, F_4, E_8$. Minuscule coweight: 0 Very stable: "Hitchin section"

Thank you!