# The decomposition theorem I: Finite fields

(Miguel González) Following: The decomp. theorem and the top. of alg. maps (de Cataldo, Migliorini)

Reading seminar on perverse sheaves, UCM, 30/01/25.

## Motivation.

Consider a "family of projective varieties"

$f : X \longrightarrow Y$

(i.e projective smooth map of vars. $/\mathbb{C}$).

Then (Deligne ~1970)

$$H^k(X, \mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(Y, \underbrace{R^q f_*(\underline{\mathbb{Q}}_X)}_{\text{semisimple local systems}}))$$

Relative $\rightarrow$ $Rf_* \underline{\mathbb{Q}}_X \cong \bigoplus R^i f_*(\underline{\mathbb{Q}}_X)[-i]$

Want a generalisation that works with singularities.

Beilinson, Bernstein, Deligne 1982 + Gabber

<u>Theo.</u> Let $f: X \longrightarrow Y$ be a proper map of complex alg. vars. Then:

$$Rf_* IC_X \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(Rf_* IC_X)[-i]$$

i-th perverse cohomology functor, $= {}^p\tau_{\geq 0}\, {}^p\tau_{\leq 0} \circ [i]$

Moreover

$$ {}^p\mathcal{H}^i(Rf_* IC_X) \cong \bigoplus_{\beta} IC_{\overline{S_\beta}}(L_\beta)$$

for $Y = \bigsqcup S_\beta$ a decomp. into finitely many disjoint locally closed smooth subv, and $L_\beta$ local systems

Hence

$$Rf_* IC_X \cong \bigoplus_a IC_{\overline{Y_a}}(L_a)[\dim X - \dim Y_a - d_a]$$

(as $S_\beta$)

In particular

$$IH^r(f^{-1}U) \cong \bigoplus_a IH^{r-d_a}(U \cap \overline{Y}_a, L_a)$$

$$(\text{if } U=Y) \quad IH^r(X) \cong \bigoplus_a IH^{r-d_a}(\overline{Y}_a, L_a)$$

<u>Examples.</u>

- $Y$ singular, $f: X \to Y$ a resolution. (proper)
  Then $IH^\cdot(Y)$ is a summand of $H^\cdot(X)$
- $X$ and $Y$ smooth, then we recover Deligne
  $$Rf_* \mathbb{Q}_X \cong \bigoplus R^i f_* \mathbb{Q}_X[-i]$$
  (Deligne)

<u>Proof of Beilinson, Bernstein, Deligne and Gabber</u>

$\to$ Uses varieties over finite fields (we will see why) Let's recall this language.

- $q$ prime, field $\mathbb{F}_q$, algebraic closure
  $\overline{\mathbb{F}_q}$ "$= \varinjlim \mathbb{F}_{q^n}$"
- Galois group $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) =$
  $=$ "$\varprojlim \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$" $=$ "$\varprojlim \mathbb{Z}/n\mathbb{Z}$" $=$
  $= \hat{\mathbb{Z}}$ pro-finite completion.

(topolg.) Generator $\rightarrow$ Inverse of $\mathrm{Fr}: \overline{\mathbb{F}_q} \to \overline{\mathbb{F}_q}$.

i.e take closure

$$t \mapsto t^q$$

- Coefficients for our sheaves
  $\longrightarrow \overline{\mathbb{Q}_\ell}$ for $\ell \neq p$, prime.
  ($\mathbb{Z}_\ell := \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Q}_\ell$ quotient field, $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)
  (same card, char and alg. cl)

· $X_0$ alg. var over $\mathbb{F}_q$
(sep. scheme of fin. type over field)

· $X$ is the base change to $\overline{\mathbb{F}_q}$.

· Sheaves: coefficients in $\overline{\mathbb{Q}_\ell}$ and on the étale topology $\rightarrow$ $\overline{\mathbb{Q}_\ell}$-sheaves.

<u>Example</u>. $X_0 = \operatorname{Spec} \mathbb{F}_q$

then $F_0$ is a finite-dim $\overline{\mathbb{Q}_\ell}$-v.s with a (cont.) action of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

we have $F_0|_{(\operatorname{Spec} \mathbb{F}_{q^n} \to \operatorname{Spec} \mathbb{F}_q)}$ = induced rep. of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_{q^n})$

and $F_0|_{\operatorname{Spec} \overline{\mathbb{F}_q}} =: F$ is just the underlying v. space. "<u>stalk</u>"

In any case, (Deligne, Grothendieck ...)

can construct analogue $D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$,

$D^b_c(X, \overline{\mathbb{Q}}_\ell)$, $\mathcal{P}(X_0, \overline{\mathbb{Q}}_\ell)$, $\mathcal{P}(X, \overline{\mathbb{Q}}_\ell)$.

Think of them as the complex case,

<u>except</u>:

<u>Key</u>: $Fr \in Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts on $\ell$-adic

sheaves over $X_0$.

New concepts appear. Let $x \in X_0(\mathbb{F}_{q^n})$ (i.e. a $Fr^n$-fixed point).

Then $F_0\big|_x$ is a $\overline{\mathbb{Q}}_\ell$-sheaf over $\operatorname{Spec} \mathbb{F}_{q^n}$

$\rightsquigarrow$ stalk $F|_x$ with $Fr^n$-action.

__Defn.__ $F_0$ is __punctually pure of weight $w \in \mathbb{Z}$__ if $\forall x \in X_0(\mathbb{F}_{q^n})$, $Fr^n$ acts with eigenvalues which are $\underbrace{\text{Weil numbers}}_{\substack{\text{alg. numbers} \\ \text{all whose conjugates} \\ \text{are of the same } |\cdot| \\ \text{via } \overline{\mathbb{Q}}_\ell \cong \mathbb{C}.}}$ of weight $q^{nw/2}$

__Defn.__

A __complex__ $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ is __pure of weight__ __$w \in \mathbb{Z}$__ if $\mathcal{H}^i(K_0)$ are punct. pure of weights $\leq w+i$ and $K_0^\vee$ has the same property.

Theo:

- <u>Relative Weil conjectures</u> (Beilinson, Bernstein, Deligne) 1982

If $f_0 : X_0 \to Y_0$ is proper of $\mathbb{F}_q$-vars, then $(f_0)_*$ sends pure complexes to pure complexes of the same weight.

New concept

"local system" $\leftrightsquigarrow$ <u>lisse $\overline{\mathbb{Q}}_\ell$-sheaf</u>.

<u>Theo (Gabber purity)</u>

If $X_0$ is connected of pure dimension $d$, then $IC_{X_0}$ is pure of weight $d$.

Finally:

<u>Theo</u>. Let $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ pure of weight $w$.

Then each ${}^p\mathcal{H}^i(K_0)$ is pure of weight $w+i$, and

$$K \cong \bigoplus_i {}^p\mathcal{H}^i(K)[-i].$$

Moreover, if $P_0 \in \mathcal{P}_M(X_0, \overline{\mathbb{Q}}_\ell)$ is pure then $P \in \mathcal{P}(X, \overline{\mathbb{Q}}_\ell)$ splits (as in the decomp. theorem) hence we get the decomp. theorem over $\overline{\mathbb{F}_q}$.

<u>Why weights and finite fields?</u>

Let $K_0, L_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ be pure of weight $w$ and $w'$ respectively.

We want to study $\mathrm{Ext}^1(K, L)$ i.e splitting behavior over $\overline{\mathbb{F}_q}$.

$$\mathrm{Ext}^1(K_0, L_0) \longrightarrow \mathrm{Ext}^1(K, L)$$

factors through $\mathrm{Ext}^1(K, L)^{Fr}$ which is pure of weight $0$.

The weight of $\mathrm{Ext}^1(K, L)$ is $1 + w' - w$.

If $w = w'$ then $\mathrm{Ext}^1(K, L)$ is weight $1$

so $\mathrm{Ext}^1(K, L)^{Fr} = 0$.

and hence extensions over $\mathbb{F}_q$ must split over $\overline{\mathbb{F}_q}$. $\square$