The Hitchin Fibration and Spectral Curves

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1 Introduction, the Hitchin fibration

We will start by recalling our object of interest: the Hitchin fibration. Let G be a complex reductive group with Lie algebra $\mathfrak{g}$ and X a compact Riemann surface of genus $g \geq 2$. We have the **moduli space of (polystable) G-Higgs bundles** $\mathcal{M}_G$, consisting of polystable pairs (E, Φ) where E is a G-bundle and $\Phi \in H^0(X, E(\mathfrak{g}) \otimes K)$, with K the canonical bundle of X.

The Hitchin fibration is defined via the G-invariants of $\mathfrak{g}$, $\mathbb{C}[\mathfrak{g}]^G$, which is, by the Chevalley Restriction Theorem, a polynomial algebra generated by elements $\{p_1, \ldots, p_r\}$

of degrees d_i. These define the fibration

$$h : \mathcal{M}(G) \to \mathcal{A} = \bigoplus_{i=1}^{r} H^0(X, K^{d_i})$$
$$(E, \Phi) \mapsto (p_1(\Phi), \ldots, p_r(\Phi))$$

In this talk we will be interested in describing the generic fibers using spectral curves. This approach works when G is a classical group, so that we can view Higgs bundles in terms of vector bundles with extra structure.

We will start with the case $G = GL(n, \mathbb{C})$ and then move to the subgroups $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, $SO(2m+1, \mathbb{C})$ and $SO(2m, \mathbb{C})$ following the original paper by Hitchin. Then we will see that it is possible to give a description for the split real form $U(m, m)$, due to Schaposnik, and we will comment on how the description generalizes for certain *cyclic Higgs bundles*.

2 $G = GL(n, \mathbb{C})$

In the case of $GL(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_\mathfrak{n} = \mathrm{Mat}_n(\mathbb{C})$ are just $n \times n$ matrices with complex entries, and the generators of the invariant polynomial ring can be taken as the $a_1, \ldots, a_n$ appearing as coefficients of the characteristic polynomial of $A \in \mathfrak{g}$:

$$\det(xI - A) = x^n + a_1(A)x^{n-1} + \cdots + a_{n-1}(A)x + a_n(A)$$

On the other hand, a $GL(n, \mathbb{C})$-Higgs bundle can be seen in terms of vector bundles, as a pair (E, Φ) where E is a holomorphic vector bundle over X and $\Phi : E \to E \otimes K$ is a holomorphic map. Thus

$$h(E, \Phi) = (a_1, \ldots, a_n)$$

where $a_i \in H^0(X, K^i)$ are the sections appearing as the coefficients of the characteristic polynomial of the map Φ.

Now, we fix such sections $a = (a_1, \ldots, a_n)$ and we want to give a description of $h^{-1}(a)$. This will be done using a curve $S \subseteq K$, called **spectral curve**, covering X and defined as follows: let $\pi : K \to X$. Then, the pullback π^*K is a bundle on K and has a *tautological section* $\lambda : K \to \pi^*K$. The spectral curve on K is given by the equation

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

(We are abusing notation by writing a_i instead of π^*a_i. The left hand side is a section of π^*K^n over K and $S \subseteq K$ is defined by its zero locus)

The curve is a ramified covering $\pi : S \to X$ of degree n. For generic a it is irreducible. Moreover, if we allow the values of a to change, we get a linear system of divisors on K

which is base point free, so for generic values of a the resulting curve S is smooth. We restrict to those a. The genus of S is $g_S = n^2(g-1)+1$ (from Riemann-Hurwitz or adjunction formula).

We want to show:

Theorem 1. *There is a bijective correspondence between holomorphic line bundles over S of degree d' and elements of the fiber $h^{-1}(a)$ of degree d, where*

$$d = d' + n(n-1)(g-1)$$

Thus, the fiber is an abelian variety (the Picard variety of S). Now we show the correspondence. First, take a line bundle $L \to S$. The vector bundle of the corresponding Higgs pair is obtained via the direct image $E = \pi_* L$. Now, given an open subset $U \subset X$ and a section $s \in H^0(\pi^{-1}(U), L)$, multiplication by the tautological section λ gives a section $s \otimes \lambda \in H^0(\pi^{-1}(U), L \otimes \pi^* K)$. In other words, we have the map

$$H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^* K)$$

or, by definition of direct image,

$$H^0(U, E) \to H^0(U, E \otimes K)$$

which translates into the vector bundle map $\Phi : E \to E \otimes K$, completing the Higgs pair. Moreover, by construction, the section λ gives the eigenvalues of Φ, and hence $\det(\lambda I - \Phi) = 0$. Since this is an irreducible polynomial (the polynomial defining S) then it is the characteristic polynomial and $h(E, \Phi) = a$.

Conversely, given the Higgs pair $(E, \Phi) \in h^{-1}(a)$, we have on S that $\det(\lambda - \pi^*\Phi) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$. Hence, outside of the ramification divisor of S, where over $x \in X$ there are n distinct eigenvalues, we have well defined one dimensional eigenspaces. These induce a line bundle $L \to S$ verifying:

$$L \subset \ker(\lambda - \pi^*\Phi) \subset \pi^* E$$

which corresponds to (E, Φ).

The relation between degrees can be obtained knowing that $\pi_* L = E$, via the formula:

$$d = \deg \pi_* L = \deg L + \deg \pi (g-1) - (g_S - 1) = d' + n(g-1) - n^2(g-1) = d' - n(n-1)(g-1).$$

□

The dimension of the fiber $h^{-1}(a)$ after fixing the degree d, which is isomorphic to $\mathrm{Jac}(S)$, is $g_S = 1 + n^2(g-1)$, that of the Hitchin base.

3 $G = SL(n, \mathbb{C})$

In this case we have $\mathfrak{g} = \mathfrak{sl}_n = \mathrm{Mat}_n^0(\mathbb{C})$ the traceless $n \times n$ matrices with complex entries. Generators of the invariant polynomial ring are still the coefficients of the characteristic polynomial, $a_2, \ldots, a_n$, which now does not have linear term.

A $SL(n, \mathbb{C})$-Higgs bundle can be seen as a standard ($GL(n, \mathbb{C})$, as above) Higgs bundle (E, Φ) with the extra conditions that $\det(E) = \mathcal{O}$ and $\operatorname{tr} \Phi = 0$.

The spectral curve $S \subset K$ is defined as in the previous case by

$$\lambda^n + a_2 \lambda^{n-2} + \cdots + a_{n-1} \lambda + a_n = 0$$

Notice that, using the already established correspondence from $GL(n, \mathbb{C})$, we can identify line bundles $L \to S$ and Higgs pairs with (E, Φ) satisfying $\operatorname{tr} \Phi = 0$. However, one has to be more precise in order to single out the line bundles that also give $\det(E) = \mathcal{O}$.

First, since $\deg E = 0$, we are restricted to line bundles of fixed degree $d' = n(n-1)(g-1)$. However, not every such line bundle will descend to an E with trivial determinant. We have to recall the construction of the **Prym variety**.

Start with a morphism of curves $f : A \to B$. This induces a homomorphism of divisors, known as the **Norm map**:

$$\mathrm{Nm} : \mathrm{Pic}(A) \to \mathrm{Pic}(B)$$

via $\mathrm{Nm}(\sum_i n_i p_i) = \sum_i n_i f(p_i)$. Its kernel defines the prym variety: $\mathrm{Prym}(A, B) = \ker \mathrm{Nm} \subseteq \mathrm{Pic}(A)$.

It was shown by Beauville, Narashiman and Ramanan that, in the context of the spectral correspondence, one has

$$\det(\pi_* L) \simeq \mathrm{Nm}(L) \otimes K^{-(n)(n-1)/2}$$

Hence, the determinant is trivial if and only if $\mathrm{Nm}(L) \simeq K^{n(n-1)/2} = \mathrm{Nm}(\pi^* K^{(n-1)/2})$ (we use that $\mathrm{Nm}(\sum n_i \pi^{-1}(p_i)) = n \sum n_i p_i$), thus if and only if $M = L \otimes \pi^* K^{-(n-1)/2} \in \mathrm{Prym}(S, X)$. The dimension of this variety is

$$\dim \mathrm{Prym}(S, X) = \dim \mathrm{Jac}(S) - \dim \mathrm{Jac}(X) = g_S - g = (n^2 - 1)(g - 1)$$

which matches that of the Hitchin base.

4 $G = Sp(2n, \mathbb{C})$

Now we turn to the case of $Sp(2n, \mathbb{C})$, the automorphisms of a $2n$-dimensional symplectic vector space $(V, \langle , \rangle)$ that preserve the symplectic form. The Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}\mathbb{C}$

consists of the endomorphisms A of the space such that $\langle Av, w\rangle + \langle v, Aw\rangle = 0$. Notice the following: if A has distinct eigenvalues $\{\lambda_i\}$, with respective eigenvectors $\{v_i\}$, one has

$$\lambda_i \langle v_i, v_j\rangle = \langle Av_i, v_j\rangle = -\langle v_i, v_j\rangle = -\lambda_j \langle v_i, v_j\rangle .$$

Because the symplectic form is nondegenerate, it follows that form some j we have $\lambda_i = -\lambda_j$. Thus the eigenvalues come in pairs $\{\pm\lambda_i\}$ and the eigenspaces are paired by the symplectic form. This also means that the characteristic polynomial is of the form

$$\det(xI - A) = x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}.$$

The polynomials $\{a_2, \ldots, a_{2m}\}$ form a basis of generators for the invariant ring.

From the above, a $Sp(2n, \mathbb{C})$-Higgs bundle is a triple $(E, \Phi, \langle,\rangle)$ where E is a holomorphic, rank $2n$ vector bundle over X, the map $\langle,\rangle : E \otimes E \to \mathcal{O}$ is a symplectic form and $\Phi : E \to E \otimes K$ satisfies $\langle \Phi v, w\rangle + \langle v, \Phi w\rangle = 0$. Moreover, the Hitchin map is still given by the coefficient sections $\{a_{2i}\}_{i=1}^n$ of the characteristic polynomial. The spectral curve $S \subseteq K$ is given by

$$\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n-2}\lambda^2 + a_{2n} = 0.$$

Similar to the previous cases, for generic a this is an irreducible, smooth curve of genus $g_S = 4m^2(g-1) + 1$ and, as before, we want to identify which line bundles $L \to S$ give $Sp(2n, \mathbb{C})$ Higgs bundles. For this, notice that the curve has an involution

$$\sigma : S \to S$$

given by $\lambda \mapsto -\lambda$, in other words, sending each eigenvalue to the opposite. We consider the quotient $\rho : S \to S/\sigma = \bar{S}$. The goal is to see that the desired line bundles L are in correspondence with points in $\mathrm{Prym}(S, \bar{S})$. Notice that such a point $U \in \mathrm{Prym}(S, \bar{S})$ is given by a divisor $D = \sum_{p\in S} n_p p$ such that $0 = \mathrm{Nm}(D) = \sum_{[p]=\{p_1,p_2\}\in\bar{S}}(n_{p_1} + n_{p_2})[p]$, hence we need that $D + \sigma D = 0$. In other words, points in the Prym variety are those with

$$\sigma^* U \simeq U^*.$$

Now, given a $Sp(2n, \mathbb{C})$-Higgs bundle $(E, \Phi, \langle,\rangle)$, and letting $L \to S$ be the corresponding line bundle obtained via eigenspaces, we have already seen that if all eigenvalues are distinct, the eigenspace of λ and $-\lambda$ are paired. Thus, the symplectic form gives a section $L^* \otimes \sigma^* L^*$ which just vanishes at the ramification locus R of $\pi : S \to X$. In other words:

$$L^* \otimes \sigma^* L^* \simeq [R]$$

Taking degrees one sees that $\deg R$ is even. Hence we can take $T \to S$ with $T^2 = [R]$. Then, we have that

$$U := L \otimes T$$

is in the Prym variety. Indeed, $\sigma^*U = \sigma^*L \otimes \sigma^*T = L^* \otimes [R]^* \otimes \sigma^*T = L^* \otimes \sigma[R]^* \otimes \sigma^*T = L^* \otimes \sigma^*T^* = L^* \otimes T^* = U^*$. We have used that σ^* fixes $[R]$ as can be seen from the first expression above.

Conversely, such a point U defines $L \to S$ with the property $L^* \otimes \sigma^* L^* \simeq [R]$. In particular, σ defines a map $L^* \to L$ which is nonvanishing away from the ramification locus. Via the direct image, this induces the symplectic form on E.

The dimension of the fiber is then $\dim \mathrm{Prym}(S, \bar{S}) = g_S - g_{\bar{S}} = g_S - (\frac{1}{2} + \frac{g(S)}{2} - n(g-1)) = n(2n+1)(g-1)$, where $g_{\bar{S}}$ can be obtained by Riemann-Hurwitz applied to $\rho : S \to \bar{S}$. It is once again the same dimension as the Hitchin base.

5 $G = SO(2n+1, \mathbb{C})$

We move to the case of $SO(2n+1, \mathbb{C})$ of orthogonal, orientation preserving, automorphisms of an $2n+1$-dimensional vector space V equipped with an inner product $\langle , \rangle$. The Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$ are the endomorphisms A with $\langle Av, w\rangle + \langle v, Aw\rangle = 0$. Since the product is nondegenerate, the same considerations as in the previous case reveal that if A has distinct eigenvalues, zero is always an eigenvalue and the others come in pairs $\pm\lambda_i$. The inner product pairs the eigenspaces for opposite eigenvalues, and the zero eigenspace with itself. The characteristic polynomial is of the form

$$\det(xI - A) = x(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}),$$

and the elements $\{a_2, \dots, a_{2n}\}$ are the basis of invariant polynomials.

A $SO(2n+1, \mathbb{C})$-Higgs bundle is a triple $(E, \Phi, \langle , \rangle)$ where E is a holomorphic, rank $2n+1$ vector bundle over X, the map $\langle , \rangle : E \otimes E \to \mathcal{O}$ is a non-degenerate symmetric bilinear form and $\Phi : E \to E \otimes K$ satisfies $\langle \Phi v, w\rangle + \langle v, \Phi w\rangle = 0$.

Notice that, since zero is always an eigenvalue, we have a well defined line subbundle $E_0 := \ker \Phi \subseteq E$. Moreover, the map $\Omega : (v, w) \mapsto (\Phi v, w)$ is a well defined skew form on E/E_0, and it is a symplectic form on

$$V := E/E_0 \otimes K^{-\frac{1}{2}}.$$

(The fact that it is non-degenerate comes from observing that $\Lambda^{2n}V \simeq \mathcal{O}$, from the knowledge that $\Lambda^{2n+1}E \simeq \mathcal{O}$, together with the fact that $\Omega^n \in \Lambda^{2n}E^* \otimes K^n \simeq E \otimes K^n$ defines an isomorphism when restricted to E_0, that is, $E_0 \simeq K^n$)

Moreover, Φ induces a map $\Phi' : V \to V \otimes K$ because it is well defined on E/E_0, as E_0 is the kernel. It also satisfies the compatibility relation with Ω. This means that from the $SO(2n+1, \mathbb{C})$ Higgs bundle we have obtained (V, Φ', Ω), a $Sp(2n, \mathbb{C})$-Higgs bundle, whose associated spectral curve is

$$\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n-2}\lambda^2 + a_{2n} = 0.$$

We already know these bundles are given by $\mathrm{Prym}(S,\bar{S})$ and since the Hitchin base of $SO(2n+1,\mathbb{C})$ matches that of $Sp(2n,\mathbb{C})$, it is of the same dimension as the base. However, it turns out that distinct (E,Φ) can result in the same (V,Φ'). So, in order to fully specify an $SO(2n+1,\mathbb{C})$-Higgs bundle in the fiber, besides an element of $\mathrm{Prym}(S,\bar{S})$ one needs to make a choice out of two options on each point in the vanishing locus of a_{2n}. Still, this gives a covering of $\mathrm{Prym}(S,\bar{S})$ so we have the same dimension.

6 $G = SO(2n,\mathbb{C})$

For the group $G = SO(2n,\mathbb{C})$ with Lie algebra $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$, which is defined exactly as in the previous case but for even dimensional vector spaces, the eigenvalues are still paired with their opposites, and the characteristic polynomial is

$$\det(xI - A) = x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}.$$

It may seem that this case can be treated in an identical way as the $Sp(2n,\mathbb{C})$ one. However, in this situation the coefficients do **not** constitute a basis for the invariant polynomials. This is due to the fact that $\det = a_{2n} = p_n^2$, where p_n is an invariant polynomial of degree n known as the *Pfaffian*. The basis of invariant polynomials is now $\{a_2, \ldots, a_{2n-2}, p_n\}$.

Hence, in the global picture, where $SO(2n,\mathbb{C})$-Higgs bundles are defined exactly as above but for even rank, we have that the Hitchin base is $\mathcal{A} = \bigoplus_{i=1}^{n-1} H^0(X, K^{2i}) \oplus H^0(X, K^n)$. Given sections $a = (a_2, \ldots, a_{2n-2}, p_n) \in \mathcal{A}$, we want to identify, as always, the fiber $h^{-1}(a)$. For this we define the spectral curve $S \subset K$ as

$$\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n-2}\lambda^2 + p_n^2 = 0$$

Now, the curve is no longer smooth at the generic fiber: instead, at the zero locus of λ, that is, at points $p \in S$ with $\lambda(p) = 0$ and hence $p_n(p) = 0$ the spectral curve has singularities (double points). There are $\deg K^n = 2n(g-1)$ of these singularities. We can work instead with the nonsingular curve obtained after resolving the singularities

$$\hat{S} \to S.$$

Since the singularities are ordinary double points, one can retrieve the genus as

$$g_{\hat{S}} = g_S - 2n(g-1) = 2n(2n-1)(g-1) + 1.$$

As before, S has the involution $\sigma : \lambda \mapsto -\lambda$, whose fixed points occur at the singular locus of S. This involution extends to an involution of $\hat{S}$ without fixed points. Now it is possible to argue exactly as in the $Sp(2n,\mathbb{C})$ case, using the ramification divisor R of $\pi : \hat{S} \to X$, to identify the fiber with $\mathrm{Prym}(\hat{S}, \hat{S}/\sigma)$.

The dimension of the fiber can be obtained by first computing $g_{\hat{S}/\sigma}$ by Riemann-Hurwitz. We use that $\sigma : \hat{S} \to \hat{S}$ does not have fixed points, so the quotient covering is unramified, and hence

$$2 - 2g_{\hat{S}} = 2(2 - 2g_{\hat{S}/\sigma}),$$

from which

$$\dim \mathrm{Prym}(\hat{S}, \hat{S}/\sigma) = g_{\hat{S}} - g_{\hat{S}/\sigma} = n(2n-1)(g-1).$$

Again, this is the dimension of the Hitchin base, half of that of the moduli space.

7 Hitchin systems associated to finite order automorphisms

Now that we have seen examples of the spectral description in the Hitchin systems for G-bundles, we are going to introduce a new class of Hitchin fibrations, associated to a complex reductive Lie group G with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a holomorphic finite order automorphism, $\theta \in \mathrm{Aut}_m(G)$. This also gives a holomorphic finite order automorphism $\theta \in \mathrm{Aut}_m(\mathfrak{g})$. Then we will see some examples of the spectral curve description in these cases.

The data of θ gives a $\mathbb{Z}/m\mathbb{Z}$-grading of the Lie algebra, $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$, where each piece is given by the weight space of θ of weight ζ^i, where ζ is a primitive m-root of unity.

We denote by $G_0 \subseteq G$ the connected subgroup associated to $\mathfrak{g}_0$. This will be reductive and act on $\mathfrak{g}_1$ (in fact in any $\mathfrak{g}_i$) by restriction of the adjoint representation. The pair $(G_0, \mathfrak{g}_1)$ is an example of a **Vinberg pair**.

Definition 1. In the above setting, a $(G_0, \mathfrak{g}_1)$-Higgs bundle is a pair (E, Φ) where E is a principal G_0 bundle and $\Phi \in H^0(E(\mathfrak{g}_1) \otimes K)$.

Remark 1. *When θ is an involution, that is, $m = 2$, by Cartan theory it defines a distinguished antiholomorphic involution $\sigma \in Conj_2(\mathfrak{g})$, in other words, a real form of $\mathfrak{g}$. A Higgs bundle for this involution then corresponds to a $G^{\mathbb{R}}$-Higgs bundle related to the representations of the fundamental group of the surface in $G^{\mathbb{R}}$, a real form of G.*

Vinberg pairs have a Chevalley-like restriction theorem, making the existence of a Hitchin system possible. Let $\mathfrak{a} \subseteq \mathfrak{g}_1$ be a **Cartan subspace**, that is, a maximal vector subspace consisting only semisimple elements and such that $[\cdot,\cdot]|_{\mathfrak{a}\times\mathfrak{a}} \equiv 0$. Define the **little Weyl group**:

$$W(\mathfrak{a}) := N_{G_0}(\mathfrak{a})/C_{G_0}(\mathfrak{a}).$$

Then $W(\mathfrak{a})$ is a finite complex group acting by reflections on $\mathfrak{c}$, and $\mathbb{C}[\mathfrak{g}_1]^{G_0} \simeq \mathbb{C}[\mathfrak{a}]^{W(\mathfrak{a})}$, so that the invariant ring is generated finitely and freely by homogeneus polynomials $\{p_1, \ldots, p_r\}$. Thus a Hitchin system is obtained in $\mathcal{M}_{(G_0,\mathfrak{g}_1)}$.

(Analogous results exist for the whole fixed point $G^\theta \subseteq G$ whose connected component at the identity is G_0, as well as its extension G_θ given by the normalizer of G^θ in G.)

Now we study spectral data for specific examples of this construction where $G = GL(n, \mathbb{C})$ in some of the situations we call *quasi-split*.

(For the *split* cases, namely the split real form of $GL(n, \mathbb{C})$, it is shown by Schaposnik that one can find the elements corresponding to this form in the generic fibers as those with order 2 in the abelian variety, where the origin is chosen to be the Hitchin section. This works for the other classical groups as well).

8 $G = U(k, k)$

Our first example is given by the quasi-split real form $U(k, k) \subset GL(2k, \mathbb{C})$. These are the automorphisms of a $2k$-dimensional vector space V equipped with an hermitian form $\langle , \rangle$ of indefinite type (k, k) that preserve the form. The Lie algebra $\mathfrak{u}(k, k)$ consists of endomorphisms A such that $\langle Av, w\rangle + \langle v, Aw\rangle = 0$. This Lie algebra is obtained as the fixed points of an antiholomorphic involution in $\mathfrak{gl}_{2k}\mathbb{C}$: such endomorphisms are given by matrices A with $-I_{k,k}\overline{A^t}I_{k,k} = A$, where $I_{k,k} = I_k \oplus -I_k$. Via this involution, as well as the compact antiholomorphic involution $\tau(A) = \overline{A^t}$, we get the Cartan decomposition:

$$\mathfrak{u}(k, k) = (\mathfrak{u}(k) \oplus \mathfrak{u}(k)) \oplus \mathfrak{m}^{\mathbb{R}}$$

which complexifies to

$$\mathfrak{gl}_{2k}\mathbb{C} = (\mathfrak{gl}_k\mathbb{C} \oplus \mathfrak{gl}_k\mathbb{C}) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ are the off-diagonal endomorphisms. Denote by $H = GL(k, \mathbb{C}) \times GL(k, \mathbb{C})$. Then, H acts on $\mathfrak{m}$ and a $U(k, k)$-Higgs bundle is defined as a pair (E, Φ) with E a principal H-bundle and $\Phi \in H^0(X, E(\mathfrak{m}) \otimes K)$. In terms of vector bundles:

Definition 2. A $U(k, k)$-Higgs bundle is a pair (E, Φ) such that $E = W_0 \oplus W_1$ is a rank $2k$ holomorphic vector bundle that splits in two rank k pieces, and $\Phi : E \to E \otimes K$ verifies $\Phi(W_i) \subseteq W_{i+1} \otimes K$, where indices are taken mod 2.

It is easy to see that an element $A \in \mathfrak{m}$ is conjugate (via the action of H) to $-A$. Thus, whenever an eigenvalue λ appears, so does $-\lambda$ and the characteristic polynomial has the form

$$\det(xI - A) = x^{2k} + a_2x^{2k-2} + \cdots + a_{2k-2}x^2 + a_{2k}.$$

The polynomial coefficients $\{a_2, \ldots, a_{2n}\}$ are a basis for the invariant polynomials $\mathbb{C}[\mathfrak{m}]^H$, and hence they provide the Hitchin map. As usual, we describe the fibers with the spectral curve $S \subseteq K$ given by the equation

$$\det(\lambda I - A) = \lambda^{2k} + a_2\lambda^{2k-2} + \cdots + a_{2k-2}\lambda^2 + a_{2k}.$$

We already know that line bundles $L \to S$ are in correspondence with standard Higgs pairs (E, Φ), and we want to determine which ones give $U(k,k)$-Higgs bundles. It turns out, as proven by L. Schaposnik, that these are precisely the line bundles L with $\sigma^* L \simeq L$, where $\sigma : S \to S$ is the already mentioned involution given by $\lambda \to -\lambda$.

The proof of this fact is as follows. Such a line bundle allows to lift σ as an involution of L over the involution in S. Then, we decompose sections $H^0(\pi^{-1}(U), L)$ into the invariant and anti-invariant parts with respect to σ. Each of those two parts descends, via the direct image, to rank k factors of E. Moreover, since the Higgs field is defined by multiplication by λ, which gets sent to $-\lambda$ via σ, it swaps invariant and anti-invariant sections and gives the desired behavior for Φ via the direct image.

Let us explain in a bit more of detail. Over a point $x \in S$ which is not a branch point of π (i.e. $\Phi|_x$ has distinct eigenvalues, coming in pairs by opposites), if we set $\pi^{-1}(x) = \{s_1, \ldots, s_{2k}\}$ we can choose a basis of $\pi_*(L)|_x = L|_{s_1} \oplus \cdots \oplus L|_{s_{2k}}$ that looks like $\{e_1, \ldots, e_k, \sigma e_1, \ldots, \sigma e_k\}$. Then the subspace of invariant elements is given in that basis by coordinates (α, α) and the subspace of anti-invariants is given by $(\alpha, -\alpha)$. This local inspection shows that globally both parts are subspaces of rank k.

One can recover the topological invariant $(d_0, d_1) = (\deg W_0, \deg W_1)$ from the spectral data by looking at the fixed points of σ, which are the $4k(g-1)$ vanishing points of a_{2n}. Because $\sigma^* L \simeq L$, the map σ lifts to an involution of the fiber $L|_x$ at such fixed points x. There are two possibilities: σ acting as $+1$ or -1. We let M_0 and M_1 count the number of points of each case.

We can get the degrees d_0 and d_1 by applying Riemann-Roch to the spaces of invariant and anti-invariant sections of L, but for this we need the respective dimensions h_0 and h_1. Regular Riemann-Roch applied to L allows us to find $h_0 + h_1$. The key remaining piece is the holomorphic Lefschetz theorem which gives $h_0 - h_1$.

We can recall this Holomorphic Lefschetz theorem: we have $\sigma : S \to S$ lifting to $\sigma' : L \to L$, and in the particular case $H^i(S, L) = 0$ for each $i > 0$. Then the trace ξ of (σ, σ') acting on sections $H^0(S, L)$ is given by

$$\xi = \sum_P \frac{\operatorname{tr} \sigma'|_P}{\det(1 - d\sigma|_P)},$$

where the sum is over σ-fixed points.

(Detail: Both in Riemann-Roch and above we do not use L itself, but rather $L \otimes L'$ where L' is a line bundle of high degree such that σ acts by $+1$ at every fixed point, this does not affect the type of each point and ensures vanishing of nonzero cohomology. The extra term $\deg L'$ gets cancelled upon solving the system).

The resulting degrees are:

$$d_j = \frac{\deg L + \alpha_{j,0} M_0 + \alpha_{j,1} M_1}{2} + (k - 2k^2)(g-1),$$

for rational coefficients $\alpha_{j,l}$ that can be precisely determined.

An interesting application for this spectral data is obtaining a bound for the **Toledo invariant** $\tau = d_0 - d_1$, which is a topological invariant governing the topology of the moduli space. Using the previous expressions, one finds

$$|\tau| \leq |2k(g-1) - M_1| \leq 2k(g-1),$$

which is the well known **Milnor-Wood inequality**.

9 Cyclic Higgs bundles of ranks $(k, k, \ldots, k)$

The above analysis can be extended for cyclic Higgs bundles of type $(k, k, \ldots, k)$. We quickly recall that these are Higgs bundles associated to a certain $\mathbb{Z}/m\mathbb{Z}$-grading of the lie algebra, $\mathfrak{g} = \mathfrak{gl}_{mk}\mathbb{C} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$, given by

$$\mathfrak{g}_i = \{A \in \mathfrak{g} : I_{k,k,\ldots,k} A I_{k,k,\ldots,k}^{-1} = \zeta^i A\},$$

where ζ is a primitive m-th root of unity and $I_{k,k,\ldots,k} = I_k \oplus \zeta I_k \oplus \cdots \oplus \zeta^{m-1} I_k$. In other words, splitting a km-dimensional vector space V into m pieces V_i which are k-dimensional, the grading in $\mathfrak{g} = \mathrm{End}(V)$ is given by $\mathfrak{g}_i = \bigoplus_j \mathrm{Hom}(V_j, V_{j+i})$.

These allow to define **cyclic Higgs bundles** as pairs (E, Φ) where E is a principal G_0-bundle, and $\Phi \in H^0(X, E(\mathfrak{g}_1) \otimes K)$. In terms of vector bundles:

Definition 3. An m-cyclic Higgs bundle of type $(k, k, \ldots, k)$ is a pair (E, Φ) such that $E = W_0 \oplus W_1 \oplus \cdots \oplus W_{m-1}$ is a rank mk holomorphic vector bundle that splits in m rank k pieces, and $\Phi : E \to E \otimes K$ verifies $\Phi(W_i) \subseteq W_{i+1} \otimes K$, where indices are taken mod m.

As before, it is easy to see that an element $A \in \mathfrak{g}_1$ is conjugate via the action of G_0 to ζA (and, in turn, to each $\zeta^i A$). Thus, if an eigenvalue λ apears, so do all the $\zeta^i \lambda$ and the characteristic polynomial looks like

$$\det(xI - A) = x^{mk} + a_m x^{mk-m} + \cdots + a_{mk-m} x^m + a_{mk}.$$

The polynomial coefficients $\{a_m, \ldots, a_{mk}\}$ give a basis for $\mathbb{C}[\mathfrak{g}_1]^{G_0}$ and provide the Hitchin map. Exactly as before, we describe the fibers with the spectral curve $S \subseteq K$ given by the equation

$$\det(\lambda I - A) = \lambda^{mk} + a_m \lambda^{mk-m} + \cdots + a_{mk-m} \lambda^m + a_{mk}.$$

Reasoning exactly as in the $m = 2$ case with the automorphism $\sigma : \lambda \to \zeta\lambda$ of the curve, that is, decomposing the sections of L into the i-invariant parts (meaning sections where σ acts by multiplication by ζ^i), one sees that the line bundles $L \to S$ that give cyclic Higgs bundles are precisely those with $\sigma^* L = L$.

If we want to do a local study as before to check that indeed each piece is of rank k, we notice that in a basis of the form $\{e_1, \ldots, e_k, \sigma e_1, \ldots, \sigma e_k, \ldots, \sigma^{m-1} e_1, \ldots, \sigma^{m-1} e_k\}$ the action of σ on coordinates is $(v_0, v_1, \ldots, v_{m-1}) \mapsto (v_{m-1}, v_0, \ldots, v_{m-2})$. Thus each i-th invariant part is given by coordinates $(\alpha, \zeta^{i(m-1)}\alpha, \ldots, \zeta^i \alpha)$.

It is posible to retrieve the degrees d_j of each W_i as before. We split the $2km(g-1)$ fixed points of σ into m types, depending on the factor ζ^i by which σ acts on $L|_x$. We count each type as $M_0, \ldots, M_{m-1}$. As before, we want the dimensions h_i of the i-th invariant sections. Regular Riemann-Roch gives $h_0 + h_1 + \cdots + h_{m-1}$. The holomorphic Lefschetz theorem applied to each power σ^i gives $h_0 + \zeta^i h_1 + \cdots + \zeta^{i(m-1)} h_{m-1}$. In turn, one gets:

$$d_j = \frac{1}{m}\left(\deg L + \sum_{l=0}^{m-1} \alpha_{j,l} M_l\right) + (k - mk^2)(g-1),$$

where the $\alpha_{j,l}$ are rational coefficients that can be precisely identified.

An interesting corollary of the spectral data occurs by examining the **Toledo invariant** that exists when the map from W_{m-1} to W_0 is identically zero. In that case, we have an **holomorphic chain** and there is an invariant

$$\tau = 2\sum_{j=0}^{m-1}\left(j - \frac{m-1}{2}\right) d_j.$$

We can examine what happens with this invariant in the cyclic case (that is, the previously mentioned map no longer needs to be zero) via the spectral data. Putting everything together one gets:

$$|\tau| \leq \frac{km(m-1)(m+1)}{3}.$$

This is the **Arakelov-Milnor inequality** which exists for holomorphic chains. This reasoning proves that in this case it holds for arbitrary cyclic Higgs bundles as well.

Bibliography / Further reading

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