Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Classification of very stable Higgs bundles

Clasificación de fibrados de Higgs muy estables

Memoria para optar al grado de doctor presentada por

Miguel González González

Directores:

Óscar García Prada, Tamás Hausel

2026

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UNIVERSIDAD COMPLUTENSE MADRID

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A mis padres.

To my parents.

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Abstract

Classification of very stable Higgs bundles

Let G be a connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$ and C a smooth complex projective curve. This thesis centres around the study of very stable G-Higgs bundles over C, which are defined as the Higgs bundles in the smooth locus of the moduli space $\mathcal{M}(G)$ that are fixed by the natural scaling action of $\mathbb{C}^\times$ and cannot be expressed as a limit point at zero of a non-constant $\mathbb{C}^\times$-orbit of nilpotent Higgs bundles. Very stable Higgs bundles were initially considered by Hausel and Hitchin [70], extending the notion of a very stable vector bundle of Drinfeld [42] and Laumon [93], with the motivation of identifying well-behaved Lagrangian subvarieties of $\mathcal{M}(G)$ on which mirror symmetry statements could be tested. Indeed, very stable Higgs bundles are in correspondence with the subset of such subvarieties which are moreover $\mathbb{C}^\times$-invariant, affine, irreducible, closed and contained in the smooth locus of $\mathcal{M}(G)$.

Our main contribution is the classification of very stable G-Higgs bundles with generically regular Higgs field, extending the above-cited work of Hausel and Hitchin which considered the case of $G = \mathrm{GL}_n(\mathbb{C})$. This classification is achieved by realising that, as in the previously mentioned case, the limiting $\mathbb{C}^\times$-orbits consisting of nilpotent Higgs bundles can be fully understood by simply considering the local modifications of the fixed point, known as Hecke transformations. The space of such transformations at a given point of C is described by an affine Springer fibre inside the affine Grassmannian of G, leading to the study of a natural $\mathbb{C}^\times$-action in said space. As a result, the dynamics of the limiting $\mathbb{C}^\times$-orbits consisting of nilpotent Higgs bundles are governed by the Bruhat order in the coweight lattice of $\mathfrak{g}$ applied to a naturally associated coweight-valued divisor recording the vanishing multiplicities of the fixed point's Higgs field. In particular, very stable Higgs bundles occur precisely when all the coefficients of the divisor are minuscule.

Besides this classification theorem, we study several aspects of the geometry of the Hitchin system restricted to the corresponding closed Lagrangian subvarieties. This restriction is a finite cover and we compute its order by finding the virtual equivariant multiplicity of the $\mathbb{C}^\times$-action on the Lagrangian subvariety. Moreover, for fixed points which are regular over the complement of a unique point of C, we describe the geometry of the cover as a pullback from a quotient of a Cartan subalgebra of $\mathfrak{g}$ by a parabolic subgroup of the Weyl group W. We also advance the program of the classification of very stable Higgs bundles with non-regular Higgs fields, establishing results which rule out the existence of very stable Higgs bundles with certain values of the Toledo invariant, a widely studied topological invariant of global nature.

The motivating mirror symmetry aspects are also present in the moduli space $\mathcal{M}_{sp}(G, \alpha)$ of strongly parabolic G-Higgs bundles. Using the aforementioned techniques, we also classify very stable strongly parabolic Higgs bundles with generically regular Higgs field by analysing the spaces of Hecke transformations. At a parabolic puncture, such space is now identified with an affine Springer fibre in an affine flag variety for G, and the dynamics are described by the Bruhat order in $\widetilde{W}$, the extended affine Weyl group of $\mathfrak{g}$. Therefore, such very stable parabolic Higgs bundles are

characterised by having minuscule multiplicity coweights outside of the parabolic punctures while, at the parabolic punctures, a naturally associated element of $\widetilde{W}$ which records both the vanishing multiplicities and the parabolic structures needs to have zero length. In the parabolic setting, a novel phenomenon is the existence of very stable Higgs bundles which are not everywhere regular yet the corresponding Lagrangian subvariety is a section of the Hitchin system. Exploiting this particularly simple behaviour we compute the dual brane for these subvarieties, which turns out to be given by a line bundle on the dual moduli space constructed as a representation of the universal parabolic structures on the universal parabolic bundle restricted to the punctures.

Keywords. Very stable Higgs bundle, affine Grassmannian, affine Springer fibre, Hecke transformation, affine flag variety.

MSC classification (2020). 14D20 (Primary), 14H60, 14H70.

Resumen

Clasificación de fibrados de Higgs muy estables

Sea G un grupo algebraico complejo semisimple y conexo con álgebra de Lie $\mathfrak{g}$ y C una curva proyectiva compleja lisa. Esta tesis se centra en el estudio de G-fibrados de Higgs muy estables sobre C, que se definen como aquellos fibrados de Higgs en la parte lisa del espacio de móduli $\mathcal{M}(G)$ que están fijos por la acción natural de $\mathbb{C}^\times$ dada por escalar el campo de Higgs y que no pueden ser expresados como punto límite cuando el parámetro tiende a cero de una $\mathbb{C}^\times$-órbita no constante de fibrados de Higgs nilpotentes. Los fibrados de Higgs muy estables fueron introducidos por Hausel y Hitchin [70] extendiendo la noción de fibrado vectorial muy estable de Drinfeld [42] y Laumon [93], motivados por la identificación de subvariedades lagrangianas de $\mathcal{M}(G)$ con buenas propiedades, en las que poder comprobar las predicciones de la simetría especular. En efecto, los fibrados de Higgs muy estables están en correspondencia con el subconjunto de tales subvariedades que además son $\mathbb{C}^\times$-invariantes, afines, irreducibles, cerradas y contenidas en la parte lisa de $\mathcal{M}(G)$.

Nuestra principal contribución es la clasificación de los G-fibrados de Higgs muy estables cuyo campo de Higgs es genéricamente regular, extendiendo el trabajo de Hausel y Hitchin previamente citado que considera el caso $G = \mathrm{GL}_n(\mathbb{C})$. Esta clasificación se realiza notando que, al igual que en el caso anterior, las $\mathbb{C}^\times$-órbitas de fibrados de Higgs nilpotentes que limitan en el punto fijo se pueden entender considerando modificaciones locales de este, conocidas como transformaciones de Hecke. El espacio de tales transformaciones en un punto dado de C se puede describir como una fibra de Springer afín contenida en la grassmanniana afín de G, motivando el estudio de una acción de $\mathbb{C}^\times$ natural en este espacio. Como consecuencia, la dinámica de las $\mathbb{C}^\times$-órbitas de fibrados de Higgs nilpotentes limitantes está gobernadas por el orden de Bruhat en el retículo de copesos de $\mathfrak{g}$ aplicado a un divisor natural asociado al punto fijo cuyos valores, que son copesos de $\mathfrak{g}$, registran la multiplicidad de anulación del campo de Higgs del punto fijo. En particular, los fibrados de Higgs muy estables ocurren cuando todos los coeficientes del divisor son minúsculos.

Además de este teorema de clasificación, estudiamos distintos aspectos geométricos del sistema de Hitchin restringido a las subvariedades lagrangianas cerradas correspondientes. Esta restricción es una aplicación finita y calculamos su orden mediante la multiplicidad virtual equivariante de la acción de $\mathbb{C}^\times$ en la subvariedad lagrangiana. Además, para aquellos puntos fijos que son regulares en toda la curva C menos en un punto, describimos la geometría de esta aplicación finita como un pullback de un cociente de una subálgebra de Cartan de $\mathfrak{g}$ por un subgrupo parabólico del grupo de Weyl W. También avanzamos en el programa de clasificar fibrados de Higgs muy estables con campo de Higgs no regular, estableciendo resultados de inexistencia de fibrados de Higgs muy estables con ciertos valores del invariante de Toledo, un invariante de naturaleza global ampliamente estudiado.

La motivación de la simetría especular está asimismo presente en el espacio de móduli $\mathcal{M}_{sp}(G, \alpha)$ de G-fibrados de Higgs fuertemente parabólicos. Usando las técnicas previamente mencionadas, clasificamos también los fibrados de Higgs fuertemente parabólicos muy estables cuyo campo de Higgs es genéricamente regular, analizando sus espacios de transformaciones de Hecke. En un punto

parabólico, este espacio está descrito por una fibra de Springer afín contenida en una variedad de banderas afín de G y la dinámica está dada por el orden de Bruhat en $\widetilde{W}$, el grupo de Weyl afín extendido de $\mathfrak{g}$. Por tanto, tales fibrados de Higgs parabólicos muy estables están caracterizados por tener copesos de multiplicidad minúsculos fuera de los puntos parabólicos y, además, en los puntos parabólicos, un elemento de $\widetilde{W}$ naturalmente asociado a las multiplicidades de anulación del campo de Higgs y la estructura parabólica en el punto tiene que tener longitud cero. En este contexto parabólico, un nuevo fenómeno es la existencia de fibrados de Higgs muy estables que no son regulares en todo C cuya subvariedad Lagrangiana asociada es una sección del sistema de Hitchin. Haciendo uso de este comportamiento especialmente sencillo calculamos la brana dual de estas subvariedades, que resulta ser un fibrado de línea en el espacio de móduli dual obtenido como una representación de las estructuras parabólicas universales en el fibrado parabólico universal restringido a los puntos parabólicos.

Palabras clave. Fibrado de Higgs muy estable, grassmanniana afín, fibra de Springer afín, transformación de Hecke, variedad de banderas afín.

Clasificación MSC (2020). 14D20 (Primaria), 14H60, 14H70.

Preface

The present document contains the author's own work except where otherwise attributed. Parts of the document have been adapted from the content in the author's publications, mainly [59, 60]. In particular:

- Chapter 1 contains material adapted from [59] in Sections 1.2, 1.4 and 1.5, and from [60] in Sections 1.1, 1.6 and 1.7.
- Chapter 2 contains material adapted from [59] in Sections 2.1 through 2.4.
- Chapter 3 contains work to appear in collaboration with Oscar García-Prada.
- Chapter 4 contains material adapted from [60] in every section.

The present document contains minor corrections and improvements with respect to the version of the thesis that was submitted and defended as part of the author's PhD studies.

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Introduction

Overview and goals

The focus of this thesis is the study and classification of **very stable Higgs bundles**, a notion introduced by Hausel and Hitchin [70] extending that of a very stable vector bundle of Drinfeld [42] and Laumon [93]. In the simplest terms, the problem involves two basic ingredients:

- A complex quasiprojective variety, the **moduli space of G-Higgs bundles** over a curve, denoted by $\mathcal{M}(G)$.
- A natural scaling $\mathbb{C}^\times$-action on $\mathcal{M}(G)$.

This $\mathbb{C}^\times$-action is *semiprojective*, meaning that any $\mathbb{C}^\times$-orbit can always be extended at zero resulting in a well defined, unique limit point in the fixed point locus. Thus, $\mathcal{M}(G)$ is partitioned

$$\mathcal{M}(G) = \bigsqcup_{\mathcal{E}\in\mathcal{M}(G)^{\mathbb{C}^\times}} W^+_{\mathcal{E}}$$

into subvarieties $W^+_{\mathcal{E}} \subseteq \mathcal{M}(G)$ defined as the loci of points that share a given $\mathcal{E} \in \mathcal{M}(G)$ as its limit.

These subvarieties are the *upward flows* from the general framework of Białynicki-Birula [11] concerning $\mathbb{C}^\times$-actions on varieties. Whenever $\mathcal{E} \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ is smooth, the corresponding $W^+_{\mathcal{E}}$ is geometrically very simple: it is isomorphic to a vector subspace of the tangent space $T_{\mathcal{E}}\mathcal{M}(G)$. However, passing to the closure $\overline{W^+_{\mathcal{E}}} \subseteq \mathcal{M}(G)$ can make this geometry more complicated, yet for some choices of $\mathcal{E}$ the upward flow $W^+_{\mathcal{E}}$ is already closed: these are the *very stable Higgs bundles*.

Question (Main goal). *Identify which Higgs bundles $\mathcal{E} \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ are very stable. In other words, classify the closed upward flows $W^+_{\mathcal{E}} \subseteq \mathcal{M}(G)^s$.*

Having introduced the main goal superficially, let us provide some motivation for its study. For this, we need to consider certain additional structures that $\mathcal{M}(G)$ possesses. The first is the **Hitchin map** [76], which is a proper, surjective morphism

$$h_G : \mathcal{M}(G) \to \mathcal{A}(G)$$

onto an affine space $\mathcal{A}(G)$, which is $\mathbb{C}^\times$-equivariant with respect to a weighted scaling action on $\mathcal{A}(G)$ with positive weights. This map allows to better understand $\mathcal{M}(G)$ through its fibres which, generically over $\mathcal{A}(G)$, are given by smooth abelian varieties [8, 38, 76].

In contrast, however, the fibre $h_G^{-1}(0)$, known as the **(global) nilpotent cone**, is extremely singular yet important from the point of view of the understanding of the topology of $\mathcal{M}(G)$: the

$\mathbb{C}^\times$-equivariance and properness of h_G implies that $\mathcal{M}(G)$ retracts to $h_G^{-1}(0)$, thus the latter encodes most topological information about the former. In turn, the topology of the projective (recall that h_G is proper) variety $h_G^{-1}(0)$ can be understood via the theory of Białynicki-Birula (with the added difficulty that $h_G^{-1}(0)$ is not smooth) by studying the $\mathbb{C}^\times$-action on $h_G^{-1}(0)$ itself. The complicated geometry of $h_G^{-1}(0)$ makes it already difficult to identify whether a given fixed point has a nontrivial upward flow on $h_G^{-1}(0)$. From this point of view, our main goal can be restated as follows.

Question (Main goal, version II). *Identify the zero-dimensional Białynicki-Birula upward flows for the $\mathbb{C}^\times$-action on $h_G^{-1}(0)$.*

In fact this version is the closest to the original definition of Drinfeld and Laumon, which were primarily interested in the study of the nilpotent cone, as well as the one by Hausel and Hitchin. However, it is equivalent to the first statement by virtue of [70, Proposition 2.14, Lemma 4.4].

Another related point of view (e.g. [68, Section 3.2]) is that of better understanding the geometry of h_G itself but, instead of making use of the usual fibrewise abelianisation explained above, doing so *horizontally* in subvarieties across many Hitchin fibres. More precisely, we mean considering the restriction

$$h_G|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \to \mathcal{A}(G).$$

Hausel and Hitchin proved [70, Lemma 4.6] that, when $\mathcal{E}$ is very stable, the previous map is simply a finite cover (more precisely it is finite, flat, generically unramified and surjective). A very interesting consequence of this is that if a given component of the fixed point locus $\mathcal{M}(G)^{\mathbb{C}^\times}$ contains a very stable Higgs bundle $\mathcal{E}$, one can compute the multiplicity of said component in $h_G^{-1}(0)$ simply by looking at the ramification index at $0 \in \mathcal{A}(G)$ of the cover given by $\mathcal{E}$ (in other words, its order). This is yet another point of view for our main problem, which comes with an interesting further question.

Question (Main goal, version III). *Identify the upward flows $W_{\mathcal{E}}^+$ such that $h_G|_{W_{\mathcal{E}}^+}$ is a finite morphism. Additionally, describe this map.*

Finally, there is a very different source of motivation for analysing very stable Higgs bundles coming from the point of view of *mirror symmetry* and *Langlands duality* for $\mathcal{M}(G)$. This concerns an entire circle of ideas started in [73] regarding different duality statements between $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$ where $G^\vee$ is the Langlands dual group of G. These statements have been linked [83] to mirror symmetry and *S-duality* of certain quantum field theories in mathematical physics, as well as to the *geometric Langlands correspondence* [3, 10], a far-reaching duality statement stemming from the Langlands program relating arithmetic with representation theory.

One of the predictions of this duality is the interchange of two different *types* of certain structures on $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$ known as **branes**, which represent boundary conditions for the dual physical theories whose targets are given by the moduli spaces. Deferring most of the details for later, at the moment it suffices to mention that there is a natural holomorphic symplectic from $\omega \in H^0(\Omega^2_{\mathcal{M}(G)})$ which extends the canonical symplectic form of the open dense subvariety $T^*\mathcal{N}(G) \subseteq \mathcal{M}(G)$ given by the cotangent bundle to the moduli space of principal G-bundles. Then, one of the two types (known as **BAA-brane**) is given by a flat connection supported on a subvariety $L \subseteq \mathcal{M}(G)$ which is Lagrangian with respect to ω. Conjecturally, mirror symmetry predicts that it should be dual to one of the other type, known as **BBB-brane**.

Very stable Higgs bundles enter the above picture by realising that, whenever $\mathcal{E} \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ is smooth, the corresponding upward flow $W_{\mathcal{E}}^+$ is precisely Lagrangian with respect to ω. This is a simple consequence of the fact that the natural $\mathbb{C}^\times$-action scales ω with weight one, as shown in [70, Proposition 2.10]. Moreover, if $\mathcal{E}$ is very stable, $W_{\mathcal{E}}^+$ is a closed Lagrangian subvariety,

isomorphic to an affine space, so up to isomorphism it only admits the trivial flat connection (for a given rank). This provides the last version of our main goal, which also comes with an additional problem.

Question (Main goal, version IV). *Identify the BAA-branes on $\mathcal{M}(G)$ that are supported on closed, $\mathbb{C}^\times$-invariant affine spaces. Additionally, study what is the corresponding dual BBB-brane.*

Our contributions towards the main goal in this thesis can be summarised as follows. First, in Theorem 2.24 we complete the classification in the locus of $\mathcal{M}(G)^{s\mathbb{C}^\times}$ given by Higgs bundles with generically regular Higgs field for any choice of semisimple complex algebraic group G. This generalises the work of Hausel and Hitchin [70, Theorem 4.16] for $G = \mathrm{GL}_n(\mathbb{C})$ (which works without changes for $G = \mathrm{PGL}_n(\mathbb{C})$) and provides a positive answer to the first part of [70, Conjecture 8.2]. We also exploit the techniques used in our classification to extend an unpublished result of Hausel [67] concerning the additional question in version III of the main goal, namely the description of the Hitchin map restricted to some of the very stable upward flows. This is the content of Section 2.5.

Then, for the remaining case of fixed points with a non-regular Higgs field, we use a topological invariant attached to every component of the fixed point locus $\mathcal{M}(G)^{s\mathbb{C}^\times}$, the **Toledo invariant** [14], to identify several components which do not contain very stable Higgs bundles at all. This is the content of Theorems 3.5 and 3.8, and generalises work for $G = \mathrm{GL}_n(\mathbb{C})$ (which again, holds identically for $G = \mathrm{PGL}_n(\mathbb{C})$) in [70, Remark 5.13] and [109].

Finally, we ask the same main question in the setting of *parabolic Higgs bundles*, namely in the moduli space $\mathcal{M}_{sp}(G, \alpha)$ of strongly parabolic G-Higgs bundles with weights α in the interior of the standard Weyl alcove. The corresponding $\mathbb{C}^\times$-action and Hitchin map h^{sp}_G share many properties with the non-parabolic ones, in particular, the previous questions are still well posed. In this setting, we prove Theorem 4.33 which completes the classification in the locus of generically regular fixed points for arbitrary semisimple group G. The behaviour for $G = \mathrm{GL}_n(\mathbb{C})$ (or $G = \mathrm{PGL}_n(\mathbb{C})$) was observed in [74, Section 9.3]. Furthermore, we identify a particularly simple class (in the sense that they constitute sections of h^{sp}_G) of the previously classified very stable upward flows and, for this class, we answer the additional question in version IV of the main goal, namely the identification of the dual brane as a line bundle over $\mathcal{M}_{sp}(G^\vee, \alpha)$. We test the proposed duality for $G = \mathrm{PGL}_n(\mathbb{C})$ and over the smooth fibres of the Hitchin map in Proposition 4.40.

This concludes a high-level overview of the thesis. In the following we give further context about all the aforementioned elements.

Higgs bundles and the Hitchin system

Let G be a connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$ and C a smooth projective complex curve of genus $g \geqslant 2$ with canonical line bundle K_C. A **G-Higgs bundle** is a pair (E, φ) where $E \to C$ is a principal G-bundle and φ, the *Higgs field*, is a section of $E(\mathfrak{g}) \otimes K_C$. Higgs bundles were introduced by Hitchin [75] as the natural object that appears when performing a dimensional reduction to a compact Riemann surface of the gauge-theoretical Yang–Mills self-duality equations in four dimensional euclidean space. More precisely, solutions to the so-called **Hitchin's equations** are given by Higgs bundles satisfying certain algebraic **stability** conditions [75, 115].

A **moduli space** $\mathcal{M}(G)$ parametrising isomorphism classes of polystable Higgs bundles can be constructed either with analytic techniques [75] as a Kähler reduction of the space of solutions to Hitchin's equations under a Hamiltonian gauge group action (a technique pioneered by Atiyah and Bott [23]), or via algebraic techniques from geometric invariant theory exploiting the stability conditions [103, 111, 116, 117]. From the algebraic construction it follows that the result has the

structure of a quasiprojective complex variety of dimension $2(g-1)\dim(G)$. The analytic construction showcases an even richer structure, as it can be upgraded to a *hyperkähler reduction* which endows the smooth locus of $\mathcal{M}(G)$ with the structure of a hyperkähler manifold: three complex structures (I, J, K) satisfying the quaternionic relations together with corresponding Kähler forms $(\omega_I, \omega_J, \omega_K)$.

The moduli space $\mathcal{M}(G)$ is also intricately related to the topology of the underlying curve C by means of the **non-abelian Hodge correspondence**. In short, Hitchin [75] (and Simpson [115] in a more general setting) realised that a given solution to his equations could be rewritten as a flat connection on a smooth principal G-bundle over C, providing (by the usual holonomy construction) a representation of the fundamental group $\pi_1(C)$ into G which is, moreover, completely reducible. Conversely, Corlette [35] and Donaldson [40] proved that any such representation endows the corresponding flat smooth principal G-bundle with a *harmonic metric*, which can be used to recast the flat connection back into a solution to Hitchin's equations. The result is a homeomorphism

$$\mathcal{M}(G) \simeq \mathcal{R}(G) := \mathrm{Hom}(\pi_1(C), G) /\!\!/ G$$

between the moduli space of G-Higgs bundles and $\mathcal{R}(G)$, the **G-character variety** of C. This extends the celebrated theorem of Narasimhan and Seshadri [101], concerning representations of $\pi_1(C)$ into a compact group, to the complex setting. The algebraic structures on both sides are completely different and, in fact, the extra complex and Kähler structures in the hyperkähler package of $\mathcal{M}(G)$ can also be understood as coming from $\mathcal{R}(G)$ via this correspondence.

Furthermore, $\mathcal{M}(G)$ is endowed with an algebraic completely integrable system (with respect to the natural algebraic symplectic form $\omega := \omega_J + i\omega_K$) which can be constructed by identifying the fibres of $E(\mathfrak{g}) \otimes K_C$ with the adjoint representation $\mathfrak{g}$ of G, and evaluating regular functions at the Higgs field (which determines a G-orbit in $\mathfrak{g}$ at each point). In other words, we can choose free generators $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, \dots, p_r]$ for the ring of regular functions on $\mathfrak{g} /\!\!/ G$, and define

$$\begin{aligned} h_G : \mathcal{M}(G) \to \mathcal{A}(G) := \bigoplus_{i=1}^{r} H^0(C, K_C^{\deg p_i}) \\ (E, \varphi) \mapsto (p_1(\varphi), \dots, p_r(\varphi)). \end{aligned}$$

This construction is also due to Hitchin [76] and it is usually referred to as the **Hitchin system**. It is a proper, surjective morphism whose generic fibres are smooth abelian varieties [8, 38, 76].

The immense importance of the Hitchin system goes beyond the theory of integrable systems and is present in different areas of mathematics, of which we now highlight two. The first one is **higher Teichmüller theory**. Hitchin realised [77] that a natural section of h_G could be constructed by globalising the construction of the *Kostant section* [89] of $\mathfrak{g} \to \mathfrak{g} /\!\!/ G$. Moreover, he showed that the image of this section in $\mathcal{M}(G)$ determines, via the non-abelian Hodge correspondence, a component in the character variety $\mathcal{R}(G)$ whose representations are all discrete, faithful, and with image contained into the split real form $G_{\mathbb{R}} \subseteq G$. Then, choosing $G = \mathrm{PGL}_2(\mathbb{C})$ results in a component consisting of Fuchsian representations of $\pi_1(C)$ into $\mathrm{PSL}_2(\mathbb{R})$: this is precisely the Teichmüller space parametrising conformal structures on the underlying topological surface of C. Consequently, the resulting components for higher rank groups were termed *higher rank Teichmüller spaces* (see e.g [128]), although those coming from Hitchin's section construction are not unique with these properties, and many more (conjecturally all) have been identified via the non-abelian Hodge theorem exploiting *Cayley correspondences* [25] which result in higher Teichmüller components when combined with recent work on Θ-positive representations [65].

The second important appearance of the Hitchin system is in Ngô's celebrated proof of the **fundamental lemma of the Langlands program** [102], in which he used a stacky version of the Hitchin system as a geometric framework on which both the arithmetic and representation-theoretic elements appearing in the lemma could be analysed.

Mirror symmetry and branes

The moduli spaces $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$, where $G^\vee$ is the Langlands dual group of G, exhibit different duality phenomena that fit in the context of **mirror symmetry** and **Langlands duality**. Classically, *mirror symmetry* is an algebro-geometric statement that connects two different (*mirror*) Calabi–Yau (smooth, or orbifold) varieties and, very roughly, relates the complex geometry of one (for example, the space of deformations of the complex structure) with the symplectic geometry (for example, the space of deformations of the Kähler metric) of the other. This mathematically surprising phenomenon was initially noticed [31] as a consequence of the intuition arising from theoretical physics, more precisely the duality of certain two-dimensional $N = (2, 2)$ supersymmetric conformal field theories. Several mathematical statements of different flavours have been extracted from these physical dualities, and they are collectively referred to as *mirror symmetry*.

The study of mirror symmetry in the setting of $\mathcal{M}(G)$ was started by Hausel and Thaddeus [73]. In their work, they showcase two different aspects. The first one is **Strominger–Yau–Zaslow (SYZ) mirror symmetry** [120] which states the underlying duality is geometrically explained by the existence of Lagrangian torus fibrations over the same base, where the generic fibres are dual tori. For $\mathcal{M}(G)$, surprisingly, this fibration is given precisely by the Hitchin system:

$$\begin{array}{ccc} \mathcal{M}(G) & & \mathcal{M}(G^\vee) \\ & {\scriptstyle h_G} \searrow \quad \swarrow {\scriptstyle h_{G^\vee}} & \\ & \mathcal{A}(G) \simeq \mathcal{A}(G^\vee) & \end{array},$$

since the bases are identified by the Killing form (which pairs $\mathfrak{g} \simeq \mathfrak{g}^\vee$), the fibres are generically abelian varieties and, over the same point, they are indeed dual. Hausel and Thaddeus prove this for $G = \mathrm{PGL}_n(\mathbb{C})$ (more generally, they prove a duality of gerbes, physically interpreted as *B-fields*, that account for the fact that some of the connected components of the fibres for $G = \mathrm{PGL}_n(\mathbb{C})$ are only torsors under these abelian varieties). Later, Donagi and Pantev [39] prove such a statement (duality of gerbes generically over $\mathcal{A}(G)$) for arbitrary semisimple G.

The second aspect initiated by Hausel and Thaddeus is **topological mirror symmetry**, that is, the agreement of certain topological invariants for $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$, in a similar fashion to the classical agreement (up to a *mirror* symmetry) of Hodge numbers for mirror Calabi–Yau varieties. In order to prevent singularities, they work with a twisted version $\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C}))$, $\mathcal{M}^d(\mathrm{PGL}_n(\mathbb{C}))$ obtained by changing the determinant line bundle of E (this can also be regarded as changing the structure group to a nonconstant group scheme $\mathcal{G}$ over C) which results in a smooth moduli space (for $\mathrm{SL}_n(\mathbb{C})$) or an orbifold moduli space (for $\mathrm{PGL}_n(\mathbb{C})$). The agreement is not of the standard Hodge numbers since the moduli spaces are non-compact: instead, it involves the (orbifold, equivariant) mixed Hodge numbers together with a *stringy* correction via a *fermionic shift* and twisting by the B-field. Hausel and Thaddeus prove the agreement for $n = 2, 3$ and later Gröchenig, Wyss and Ziegler [64] proved it for all values of n using arithmetic techniques. After that, Maulik and Shen [96] provided another proof with different techniques based on Ngô's work [102]. For other choices of G, the singularities of the resulting moduli spaces (which now are not so easy to *twist* away without considering parabolic Higgs bundles or different twisting line bundles) make the problem more complicated and, to the author's knowledge, it is still open at the time of writing this thesis.

Besides this proposal and the subsequent developments, an important breakthrough by Kapustin and Witten [83] provided a physical explanation for the mirror symmetry features of $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$ as coming from a dimensional reduction to a compact Riemann surface of the physical **S-duality** in four-dimensional $N = 4$ supersymmetric Yang–Mills theories. This is a kind of duality

that generalises electromagnetic duality [129] and changes the symmetry group G to its Langlands dual $G^\vee$. Moreover, the dimensional reduction results in a duality of two-dimensional $N = (4,4)$ supersymmetric theories whose target spaces are precisely $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$.

One of the core aspects of mirror symmetry in physics is the interchange of the structures determining boundary conditions on each of the dual theories. These structures are known as **branes** (also *D-branes*, or *Dirichlet branes*). For $\mathcal{M}(G)$, Kapustin and Witten identify four different special types of branes depending on how they interact with the hyperkähler structure of the moduli space (physically, the different types stem from whether they are preserved by the topological twist of the *A-model* or of the *B-model* in each of the directions of the sphere of complex structures). One type is **BBB-branes**, given by hyperholomorphic subvarieties with a hyperholomorphic connection (i.e. a connection on a vector bundle which is holomorphic with respect to the three complex structures I, J, K) supported on them. S-duality interchanges this type with **BAA-branes**, which are not completely described but include flat connections supported on I-holomorphic subvarieties which are Lagrangian with respect to $\omega = \omega_J + i\omega_K$. Rotating the role of (I, J, K) one can analogously consider **ABA-branes** and **AAB-branes**, each of which is exchanged with another one of the same type as itself.

Interestingly, from this interchange of branes (which in the setting of compact Calabi–Yau varieties could be formulated mathematically as Kontsevich's **homological mirror symmetry** [87]), Kapustin and Witten also deduce the geometric Langlands correspondence [3, 10] via physical arguments.

Motivated by this intuition, there have been ongoing efforts to identify branes of each of these types in $\mathcal{M}(G)$, propose the corresponding dual (or *mirror*) brane and test the duality. Regarding the first matter, natural examples of BBB-branes include branes with full support obtained from evaluating universal bundles (whenever they exist) at G-representations and restricting them to a point $c \in C$ (see e.g. [70, Section 7]), as well as fixed points by involutions of the character variety $\mathcal{R}(G)$ coming from holomorphic involutions of the group G and of the base curve C. For BAA-branes, natural examples include the images of $\mathcal{A}(G)$ via Hitchin sections [76], the fibres of the Hitchin map, and our main example of interest: Białynicki-Birula upward flows $W^+_{\mathcal{E}}$ of the natural $\mathbb{C}^\times$-action on $\mathcal{M}(G)$. Moreover, there are those given by involutions of $\mathcal{R}(G)$ coming from antiholomorphic involutions of G and holomorphic involutions of C. Instead, choosing antiholomorphic involutions for C and holomorphic for G produces ABA-branes, and taking both types of involutions to be antiholomorphic produces AAB-branes. The general study of involutions concerning types BAA and BBB was carried out by García-Prada and Ramanan [56], while for ABA and AAB there is work of Baraglia and Schaposnik [6] as well as a general description by Biswas and García-Prada [20]. More generally, supports for BBB-branes can be obtained as fixed point loci of a wide class of finite group actions on $\mathcal{M}(G)$, thoroughly studied in [7].

Of course, understanding the duality between branes is a potential source for finding more examples by taking the existing ones and computing their dual. This brings us to the other question of mathematically understanding the mechanism by which branes are exchanged. Over the generic locus of the Hitchin base (corresponding to smooth fibres of h_G), Donagi and Pantev proved [39, Corollary 5.5] that a fibrewise Fourier–Mukai transform (using the Poincaré line bundle as a kernel) establishes an equivalence of categories between bounded, derived categories of coherent sheaves

$$D^b(\mathcal{M}(G)) \simeq D^b(\mathcal{M}(G^\vee)).$$

From the point of view of mirror symmetry, as explained more in detail in Section 1.3, this is the mechanism that can be used to test the duality of (BAA- and BBB-) branes, mapping a given brane restricted to the generic locus of $\mathcal{A}(G)$ to its dual. This has been tested in several situations: for example, [70, Section 6] shows that certain very stable upward flows for $G = \mathrm{GL}_n(\mathbb{C})$ (or $\mathrm{PGL}_n(\mathbb{C})$) are interchanged with BBB-branes of the form $\rho(\mathbb{E}|_{\mathcal{M}(G)\times c})$ constructed from the universal bundle restricted to a point $c \in C$ and evaluated at a representation ρ.

Very stable Higgs bundles

As explained above, one source of BAA-branes is given by the fibres of the Białynicki-Birula fibration for the natural scaling $\mathbb{C}^\times$-action on $\mathcal{M}(G)$, where $\lambda \in \mathbb{C}^\times$ maps

$$(E, \varphi) \mapsto (E, \lambda\varphi).$$

Given a fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$, its **upward flow**

$$W^+_{(E,\varphi)} := \left\{ (E', \varphi') \in \mathcal{M}(G) : \lim_{\lambda \to 0} (E', \lambda\varphi') = (E, \varphi) \right\} \subseteq \mathcal{M}(G)$$

turns out to be a holomorphic Lagrangian subvariety whenever (E, φ) is a smooth point. Following Hausel and Hitchin [70], we say that a smooth fixed point (E, φ) is **very stable** if $W^+_{(E,\varphi)}$ contains no nilpotent G-Higgs bundles other than (E, φ) itself, which is equivalent to $W^+_{(E,\varphi)} \subseteq \mathcal{M}(G)$ being closed. Nilpotent here means that the Higgs field takes values in nilpotent G-orbits of $\mathfrak{g}$ everywhere over C or, equivalently, that the Higgs bundle lies in $h_G^{-1}(0)$. A smooth fixed point which is not very stable is said to be **wobbly**.

Very stable and wobbly Higgs bundles have been previously studied in the literature. For $G = \mathrm{GL}_n(\mathbb{C})$, the original work of Laumon [93] considers the case of very stable vector bundles, that is, when $\varphi = 0$. In this work, it is shown that the very stable locus is open and dense and, recently, Pal [104] announced that the wobbly locus is of pure codimension one. In this same setting, Pauly and Peón-Nieto characterised [107] very stable vector bundles in terms of the properness of the Hitchin map restricted to their upward flows. At the other extreme, the work of Hausel and Hitchin [70] considers fixed points where φ is generically regular and fully classifies them in terms of a natural divisor over C associated with such a fixed point. In the same setting, [61] gives a similar result restricted to the fixed point locus of the involution given by the action of $\lambda = -1$, which is related [56] to Higgs bundles for the real structure group $U(n, n)$ and $U(n, n+1)$. For non-regular, non-zero Higgs fields there is work of Peón-Nieto [109] establishing the wobbliness of many of the components where $E = E_0 \oplus E_1$ and $\varphi : E_0 \to E_1 \otimes K_C$. Finally, for general G, the case with zero Higgs field was studied by Zelaci [133].

As our main goal is to classify very stable Higgs bundles in $\mathcal{M}(G)$, first we have to recall how the fixed point locus $\mathcal{M}(G)^{\mathbb{C}^\times}$ can be described, as the classification will differ depending on the *type* of fixed point. Fixed points of the $\mathbb{C}^\times$-action can be classified by considering $\mathbb{Z}$-gradings of the Lie algebra, that is, vector space direct sum decompositions

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. In this situation there exists an element $\zeta \in \mathfrak{g}_0$ such that $\mathfrak{g}_j = \ker(\mathrm{ad}(\zeta) - j\,\mathrm{Id})$. Fixed points of the $\mathbb{C}^\times$-action then correspond [14, Section 4.2] (see also [113]) to G-Higgs bundles (E, φ) with a reduction of structure group E' of E from G to the centraliser $G_0 := C_G(\zeta)$ and such that φ belongs to $E'(\mathfrak{g}_j) \otimes K_C$ for some $j \neq 0$. Rescaling the grading and passing to a subalgebra we will typically assume that $j = 1$. The **type** of the fixed point refers to the conjugacy class of the parabolic $\bigoplus_{j \geqslant 0} \mathfrak{g}_j \subseteq \mathfrak{g}$. Such gradings and parabolic subalgebras are completely classified (e.g. [62, Section 3.5]) in terms of labellings of the Dynkin diagram of $\mathfrak{g}$.

Classification with regular Higgs field

The classification of very stable fixed points $(E, \varphi) \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ where φ lies generically in a regular (nilpotent) G-orbit in $\mathfrak{g}$ turns out to be entirely local in nature, determined by how the Higgs field looks like around the finitely many points in C where it leaves the regular nilpotent orbit. This is similar to the case of $G = \mathrm{GL}_n(\mathbb{C})$ as noticed by Hausel and Hitchin [70] however,

unlike that case, depending on the group G not every component of the fixed point locus $\mathcal{M}(G)^{\mathbb{C}^\times}$ will possess very stable Higgs bundles.

In terms of $\mathbb{Z}$-gradings, the type of such a fixed point is what we call *Borel type* (since the corresponding parabolic is a Borel subalgebra). More precisely, the degree of a root space is given by the height of the root, so that $G_0 = T$ is a maximal torus and $\mathfrak{g}_1 = \bigoplus_{j=1}^r \mathfrak{g}_{\beta_j}$ is the sum of a choice of simple root spaces for the action of said torus. Therefore, we recover sections $\varphi_j \in H^0(E(\mathfrak{g}_{\beta_j}) \otimes K_C)$ of line bundles which are moreover nonzero, as otherwise φ would not be generically regular.

We encode the relevant information in a divisor $\mu(E,\varphi)$, the **multiplicity divisor**, valued in the lattice of dominant coweights of $\mathfrak{g}$ and defined by the property

$$\beta_j(\mu(E,\varphi)|_c) = \mathrm{ord}_c(\varphi_j)$$

i.e. its pairing with a simple root β_j recovers the divisor of zeroes of φ_j. Then, there is a natural notion of **reduced** divisor which is different from the *naive* reduction of the divisor of each φ_j. Instead, we say that $\mu(E,\varphi)$ is reduced whenever all the coefficients are *minuscule* coweights of $\mathfrak{g}$, meaning that their pairing with the highest root equals one. In these terms, we can cleanly state one of the main classification theorems in this thesis:

Theorem (Theorem 2.24). *Let (E,φ) be a smooth $\mathbb{C}^\times$-fixed point of Borel type with multiplicity divisor $\mu(E,\varphi)$. Then, (E,φ) is very stable if and only if $\mu(E,\varphi)$ is reduced.*

The phenomenon observed by Hausel and Hitchin [70, Corollary 4.19] that every fixed point component of Borel type for $G = \mathrm{GL}_n(\mathbb{C})$ (and, completely analogously $G = \mathrm{PGL}_n(\mathbb{C})$) contains very stable Higgs bundles is, under the lens of the previous result, a consequence of the fact that minuscule coweights generate the coweight lattice of any type A simple Lie algebra. However, this is far from true in other types of simple Lie algebras, with the most extreme example being in types E_8, F_4 and G_2 where the span of the minuscule coweights is trivial: our result then implies that only everywhere-regular Higgs bundles are very stable for these groups (assuming generic regularity to begin with).

As previously mentioned, the classification is local in nature: every fixed point of Borel type (E,φ) is generically regular and, in fact, away from finitely many points of C it is isomorphic to an everywhere regular $(E^0,\varphi^0) \in \mathcal{M}(G)^{\mathbb{C}^\times}$. We say that (E,φ) is **Hecke transformation** of (E^0,φ^0) (more precisely, assuming that $G = G_{\mathrm{ad}}$, (E,φ) can be obtained by a *sequence* of Hecke transformations at the different points of C starting from (E^0,φ^0)).

Moreover, not only can fixed points of Borel type be related by performing Hecke transformations, but also the upward flow of (E,φ) inside the nilpotent cone $h_G^{-1}(0)$ can be described entirely in terms of Hecke transformations of (E^0,φ^0) (again, whenever $G = G_{\mathrm{ad}}$, otherwise some minor adaptations are needed). In other words, all the upward flow dynamics in the fixed point components of Borel type stay confined within components of the same type in the sense that, if (E,φ) were wobbly, its upward flow would limit to other fixed points of Borel type only.

This leads naturally to the study of a natural $\mathbb{C}^\times$-action (cf. Proposition 2.18) on the space of Hecke transformations of (E,φ) at a given point $c \in C$, which is identified with an affine Springer fibre in the affine Grassmannian $\mathrm{Gr}_G = G((z))/G[[z]]$. Understanding the upward flows for this $\mathbb{C}^\times$-action allows to deduce the classification theorem, and this local problem is much better understood. We remark that, for $G = \mathrm{GL}_n(\mathbb{C})$, again due to minuscule coweights generating the whole lattice, only the *elementary* Hecke transformations i.e. those given by finite Grassmannians $\mathrm{Gr}(k, E|_c)$ of the fibre (which embed naturally into $\mathrm{Gr}_{\mathrm{GL}_n(\mathbb{C})}$ as closed subvarieties) were needed, allowing to solve the problem with classical Hecke modifications in [70].

In this setting, whenever (E,φ) is very stable, we can also shed some light on how the finite map $h_G|_{W^+_{(E,\varphi)}}$ looks like. First, we can compute its ramification index over 0 (in other words, its order)

by computing the **virtual equivariant multiplicity** $m_{(E,\varphi)}(t) \in \mathbb{Z}((t))$ introduced by Hausel and Hitchin [70, Section 5] of any fixed point (E,φ) of Borel type and evaluating it at $t = 1$. In the simplest example of fixed points where $\mu(E,\varphi)$ is supported at a single point, the resulting equivariant multiplicity only depends on the nontrivial minuscule coweight in $\mu(E,\varphi)$ and it agrees with the *Dynkin polynomial* of the representation of $\mathfrak{g}^\vee$ given by said coweight, a representation-theoretic invariant (see e.g. [105]). In particular, the resulting order (after evaluating at $t = 1$) is the Euler characteristic of the flag variety G/P where $P \subseteq G$ is the parabolic subgroup corresponding to the minuscule coweight.

Furthermore, for those simplest very stable upward flows (with $\mu(E,\varphi)$ supported at a point) these techniques allow us to generalise an unpublished result of Hausel [67] showing that the corresponding finite map is given as a pullback of the quotient map $\mathfrak{t}/W_\mu \to \mathfrak{t}/W$ where W_μ is the parabolic subgroup of W corresponding to the minuscule coweight. This projection map can also be interpreted in terms of the projection $\operatorname{Spec} H_G(G/P) \to \operatorname{Spec} H_G(\mathrm{pt})$ in equivariant cohomology.

Towards the non-regular case

For fixed points (E,φ) that are not of Borel type, the situation is no longer governed entirely by local phenomena, and different difficulties arise. For example, it is no longer true that there is a unique model of fixed point (E^0,φ^0) such that (E,φ) is isomorphic to it away from finitely many points. Besides, different fixed point types enter the picture as the closure of the upward flow for (E,φ) may contain fixed points whose type is not the same as that of (E,φ). Crucially, this means that the upward flows can no longer be modelled via Hecke transformations (as these are local and preserve the fixed point type).

Instead, our approach in this thesis is to showcase how global phenomena can affect the classification, by using topological invariants that can be attached to a whole component of the fixed point locus in order to produce elements in the upward flow of the component in $h_G^{-1}(0)$. In other words, we seek to exclude entire components of the fixed point locus from having very stable Higgs bundles by considering these global invariants.

More precisely, we focus on the **Toledo invariant** of the fixed point (E,φ) introduced by Biquard, García-Prada, Collier and Toledo [14]. This is a rational number $\tau(E,\varphi)$ which is constrained by the *Arakelov–Milnor inequality*

$$-(2g-2)\operatorname{rank}_T(G_0,\mathfrak{g}_1) \leqslant \tau(E,\varphi) \leqslant 0$$

where $\operatorname{rank}_T(G_0,\mathfrak{g}_1)$ is a number attached to the $\mathbb{Z}$-grading called its *Toledo rank*.

While this invariant can be defined for any type of fixed point, there is a specific class of types in which it is essentially *unique*, governing most of the relevant properties of the fixed point component. These are the gradings which we call of *Hermitian type:*

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

They correspond to parabolic subalgebras of $\mathfrak{g}$ with abelian nilpotent radical. The name comes from the fact that there is an associated real form $G_\mathbb{R} \subseteq G$ of *Hermitian type*, meaning that the associated symmetric space $G_\mathbb{R}/H_\mathbb{R}$ (where $H_\mathbb{R} \subseteq G_\mathbb{R}$ is a maximal compact subgroup) is Hermitian, in such a way that G_0 agrees with the complexification of $H_\mathbb{R}$ and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is the eigenspace decomposition of the complexification of the tangent space at the identity class in $G_\mathbb{R}/H_\mathbb{R}$ under the almost complex structure.

Fixed point components of Hermitian type are contained within the locus of $G_\mathbb{R}$-Higgs bundles in $\mathcal{M}(G)$, meaning that under the non-abelian Hodge correspondence they result in representations whose images lie in $G_\mathbb{R}$. These Higgs bundles and their moduli spaces $\mathcal{M}(G_\mathbb{R})$ have their own Toledo invariant, which extends the aforementioned one and significantly predates it, with origins in the

work of Toledo [125] studying the $\mathrm{PSU}(1,n)$-character variety. $G_{\mathbb{R}}$-Higgs bundles and their moduli spaces have been thoroughly studied [1,2,16,25,26,28,29] and play a key role in higher Teichmüller theory, as higher Teichmüller spaces can be obtained from the components in $\mathcal{M}(G_{\mathbb{R}})$ whose Toledo invariant attains its lower bound.

Our results for fixed points of Hermitian type exploit the Toledo invariant in two different ways. The first one concerns the *regular type* (also known as *tube type* in relation to the geometry of $G_{\mathbb{R}}/H_{\mathbb{R}}$) which occurs when the G_0-centraliser G_0^e of a generic element in $e \in \mathfrak{g}_1$ is reductive. In this setting, the locus of components with Toledo invariant $\tau(E,\varphi) = -(2g-2)\operatorname{rank}_T(G_0,\mathfrak{g}_1)$ attaining the lower bound is isomorphic to the moduli space of principal G_0^e-bundles via the *Cayley correspondence*. When $G_{\mathbb{R}}$ is not the split real form, this can be exploited by constructing nonzero, nilpotent K_C^2-twisted Higgs fields for the given G_0^e-bundles which then, through the Cayley correspondence, imply the wobbliness of (E,φ), resulting in the following.

Theorem (Theorem 3.5). *Let $(E,\varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ be a fixed point of Hermitian type whose associated Hermitian grading is regular and such that $G_{\mathbb{R}} \subseteq G$ is not split. Suppose that $\tau(E,\varphi) = -\tau_{\max} = -(2g-2)\operatorname{rk}_T(G_0,\mathfrak{g}_1)$. Then, (E,φ) is wobbly.*

The non-splitness constraint is not very restrictive: it only excludes $G_{\mathbb{R}} = \mathrm{Sp}_{2n}(\mathbb{R})$ and few other low rank cases.

The second situation occurs in the *non-regular* (or *non-tube*) type. When the Toledo invariant is large in absolute value, the Higgs field φ can be deformed into the upward flow by adding a section of $E(\mathfrak{g}_{-1}) \otimes K_C$ in such a way that the result is nilpotent.

Theorem (Theorem 3.8). *Let (E,φ) be a stable fixed point of Hermitian type with Toledo invariant $\tau(E,\varphi)$. There exists an explicit (see Theorem 3.8) value $\tau' \geqslant -(2g-2)\operatorname{rank}_T(G_0,\mathfrak{g}_1)$ depending on the $\mathbb{Z}$-grading of $\mathfrak{g}$ such that*

$$\tau(E,\varphi) < \tau'$$

implies that (E,φ) is wobbly.

The above value is strictly larger than $-(2g-2)\operatorname{rank}_T(G_0,\mathfrak{g}_1)$ precisely for non-regular (non-tube type) gradings, resulting in components without very stable Higgs bundles. For some specific gradings all the components of Hermitian type are covered, while for others the situation with $\tau(E,\varphi) \geqslant \tau'$ is yet to be determined. We also expect this technique to work for more general types of gradings, but in that case a better understanding of the other topological invariants besides the Toledo invariant is needed.

Parabolic Higgs bundles

Now fix a divisor $D = c_1 + \cdots + c_s$ of distinct punctures in C. The non-abelian Hodge correspondence can be generalised (as initiated by Simpson [114]) to account for the topology of the punctured surface $C \smallsetminus D$, giving rise to the theory of **parabolic Higgs bundles**. The construction depends on the choice of *parabolic weights* $\alpha = (\alpha_1,\ldots,\alpha_s)$, modelling the compact part of the monodromy of the corresponding representation around each puncture c_i. Here $\alpha_i \in \mathfrak{t}_{\mathbb{R}}$ lies in the real span of the coweight lattice inside a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. In this thesis we assume for simplicity that the weights lie in the interior of the standard Weyl alcove, although we explain how many of the steps can be carried out with more general choices of weights.

With these assumptions, a parabolic G-Higgs bundle over (C,D) is given by a triple (E,φ,Q) where E is a principal G-bundle over C, φ is a section of $E(\mathfrak{g}) \otimes K_C(D)$ and $Q = (Q_1,\ldots,Q_s)$ is a choice of *parabolic structures* $Q_i \in E|_{c_i}/B$, where $B \subseteq G$ is a fixed Borel subgroup. The Higgs field

φ is furthermore required to be compatible with said parabolic structures in a natural way. For such an object, there are notions of stability depending on α which allow a quasi-projective moduli space $\mathcal{M}(G,\alpha)$ to be constructed [12, 15, 86, 131]. As anticipated above, it satisfies a non-abelian Hodge correspondence

$$\mathcal{M}(G,\alpha) \simeq (\mathrm{Hom}(\pi_1(C \smallsetminus D), G) /\!\!/ G)_\alpha\,,$$

that identifies it (non-algebraically) with the character variety for $C \smallsetminus D$ with fixed compact part of the monodromy given by α. This extends the theorem of Mehta and Seshadri [97] relating stable parabolic vector bundles and unitary representations of the fundamental group of $C \smallsetminus D$.

The moduli space $\mathcal{M}(G,\alpha)$ has the structure of a **Poisson variety**, with hyperkähler symplectic leaves determined by fixing the whole monodromy of the representation around the punctures (not just the compact part) [15, Proposition 7.1], [24, 94, 95]. We will focus on a particular leaf where the monodromies are simply required to be compact, called the moduli space of **strongly parabolic** Higgs bundles

$$\mathcal{M}_{sp}(G,\alpha) \subseteq \mathcal{M}(G,\alpha).$$

This symplectic leaf behaves in many ways similarly to the usual moduli space $\mathcal{M}(G)$. For example, it is endowed with a Hitchin system

$$\begin{aligned} h^{sp}_G : \mathcal{M}_{sp}(G,\alpha) &\to \mathcal{A}_{sp}(G) := \bigoplus_{i=1}^{r} H^0(C, K_C^{d_i}((d_i-1)\cdot D)) \\ (E,\varphi,Q) &\mapsto (p_1(\varphi),\dots,p_r(\varphi)). \end{aligned}$$

which is a proper, surjective (onto $\mathcal{A}_{sp}(G)$, note that it is not just the *naive* $K(D)$-twisted version of the usual Hitchin base) integrable system. For our choice of weights, its generic fibres are also smooth abelian varieties by means of a BNR correspondence [57, 121], with the main difference that the spectral curves are always totally ramified over the parabolic punctures.

In this setting, there are many ongoing efforts to identify mirror symmetry phenomena for $\mathcal{M}_{sp}(G,\alpha)$ and $\mathcal{M}_{sp}(G^\vee,\alpha^\vee)$ (notice that the spaces of weights can be identified via $\mathfrak{t} \simeq \mathfrak{t}^\vee$). These stem from the physical intuition [66] of including *surface operators* in the starting four dimensional supersymmetric $N = 4$ Yang–Mills theory, corresponding to allowing defects on the fields supported at two-dimensional surfaces. Mathematically, already the original announcement of Hausel and Thaddeus [72] considered parabolic punctures, as choosing the weights α generically guarantees that the resulting moduli spaces are smooth or of orbifold type without the need of twisting the determinant line bundle that arose in the non-parabolic setting.

For example, there are results regarding SYZ mirror symmetry

$$\begin{array}{ccc} \mathcal{M}_{sp}(G,\alpha) & & \mathcal{M}_{sp}(G^\vee,\alpha^\vee) \\ & {\scriptstyle h^{sp}_G} \searrow \quad \swarrow {\scriptstyle h^{sp}_{G^\vee}} & \\ & \mathcal{A}_{sp}(G) \simeq \mathcal{A}_{sp}(G^\vee) & \end{array}$$

for $G = \mathrm{SL}_n(\mathbb{C})$ by Biswas and Dey [18], extended by Shen [112] and Su, Wang and Wen [122] to more general (non-algebraically closed) fields. There has also been recent work of Wang, Wen and Wen [127] towards the duality for types B and C.

The aforementioned works of Shen and Su–Wang–Wen also establish the topological mirror symmetry for $G = \mathrm{SL}_n(\mathbb{C})$, as an agreement of E-polynomials (where that of $G = \mathrm{PGL}_n(\mathbb{C})$ is again the *stringy* version accounting for the orbifold singularities and twisted by a gerbe). As previously noted, in the parabolic setting this statement holds for the moduli spaces without any further

twistings since choosing generic weights guarantees their smoothness (up to orbifold singularities). Prior to this work, Gothen and Oliveira [63] already established the topological mirror symmetry statement for $n = 2, 3$. They noticed that the twisting by a gerbe is not strictly necessary and they do not consider it. The reason is that, in contrast with the non-parabolic setting, there always exist choices of sections of the Hitchin map contained in any given connected component of $\mathcal{M}_{sp}(G, \alpha)$, so over the generic locus of $\mathcal{A}_{sp}(G)$ the gerbe can be trivialised.

Scaling the Higgs field by a nonzero complex number defines a $\mathbb{C}^\times$-action on $\mathcal{M}_{sp}(G, \alpha)$ with similar properties to the one on $\mathcal{M}(G)$. In particular, we can consider upward flows $W^+_{(E,\varphi,Q)} \subseteq \mathcal{M}(G, \alpha)$ and they constitute holomorphic Lagrangian subvarieties, becoming the support of BAA-branes. We define **very stable** and **wobbly** parabolic Higgs bundles similarly, and the classification problem becomes relevant in this setting as well. In the literature, very stable parabolic vector bundles (in other words, the case with $\varphi = 0$) have been studied by Peón-Nieto [108], and the behaviour in the case of generically regular φ for $G = \mathrm{GL}_n(\mathbb{C})$ was observed by Henke [74].

The first step in this problem is of course to describe and understand the fixed point locus $\mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$. While there are some references in the literature [37] mainly concerned with the vector bundle setting (i.e. $G = \mathrm{GL}_n(\mathbb{C})$), we study the characterisation in terms of $\mathbb{Z}$-gradings of the Lie algebra $\mathfrak{g}$.

Theorem (Theorem 1.92). *A parabolic G-Higgs bundle $(E, \varphi, Q) \in \mathcal{M}(G, \alpha)$ is fixed by the $\mathbb{C}^\times$-action if and only if there is a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ such that*

- *There is a reduction $E_\sigma \subseteq E$ of structure group to G_0.*
- *The Higgs field is a section of $E_\sigma(\mathfrak{g}_k) \otimes K_C(D) \subseteq E(\mathfrak{g}) \otimes K_C(D)$, for some $k \neq 0$, which is compatible with the parabolic structures.*
- *The parabolic structures satisfy*

$$Q_i \in \bigsqcup_{w\in W_{G_0}\backslash W} \left((E_\sigma)\,|_{c_i} \cdot w \cdot B\right)/B \subseteq E|_{c_i}/B.$$

In the above, we let $W := N_G(T)/T \supseteq N_{G_0}(T)/T =: W_{G_0}$ denote the respective Weyl groups. Having identified the structure of fixed points, we focus on the situation where φ is generically regular. As for $\mathcal{M}(G)$, this implies that $G_0 = T$ is abelian and the Higgs field takes values in $\mathfrak{g}_1$ the sum of the simple root spaces, therefore we may once again extract a multiplicity divisor $\mu(E, \varphi)$ which does not depend on Q (although its possible values are constrained by it) by looking at the components on each line bundle $E(\mathfrak{g}_{\beta_j}) \otimes K_C(D)$. However, we now have the additional information of the parabolic structures, which according to the theorem must satisfy

$$Q_i \in \bigsqcup_{w\in W} \left((E_\sigma)\,|_{c_i} \cdot w \cdot B\right)/B \simeq W \subseteq E|_{c_i}/B.$$

Therefore, we can attach at each c_i an element $u_{c_i}(E, \varphi, Q) \in W$, and combine everything in the **twisted multiplicity divisor** $w(E, \varphi, Q)$ taking values in the extended affine Weyl group $\widetilde{W}$ of $\mathfrak{g}$. This divisor is defined by

$$w(E, \varphi, Q)|_c := (-\mu(E, \varphi)|_c) \cdot u_c(E, \varphi, Q).$$

It turns out that there is also a notion of $w(E, \varphi, Q)$ being reduced: its coefficients at parabolic punctures belong to $\widetilde{W}$, while at points $c \in C \smallsetminus D$ they are antidominant coweights of $\mathfrak{g}$, which can be regarded as elements of the double quotient $W\backslash\widetilde{W}/W$. Both groups are examples of parabolic quotients of (quasi-) Coxeter groups, so they are endowed with a natural *Bruhat order*, and we say that $w(E, \varphi, Q)$ is reduced whenever all its coefficients are minimal under said order (for $c \in C \smallsetminus D$

this is equivalent to the coweight being minuscule, as for $\mathcal{M}(G)$). Then, we have a classification result in the same terms as the non-parabolic one.

Theorem (Theorem 4.33). *Let $(E,\varphi,Q)\in\mathcal{M}_{sp}(G,\alpha)^{s\mathbb{C}^\times}$ be a smooth fixed point of Borel type. Then, (E,φ,Q) is very stable if and only if $w(E,\varphi,Q)$ is reduced.*

The strategy of the proof is similar in the parabolic setting, by relying in the local nature of the problem: again, at least when $G=G_{\mathrm{ad}}$ (otherwise some adaptations can be made), every fixed point (E,φ,Q) of Borel type can be obtained by a sequence of Hecke transformations of an everywhere regular fixed point (E^0,φ^0,Q^0), whose parabolic structure is uniquely determined by the regularity of φ^0. The main difference is that the space of Hecke transformations at a parabolic puncture is no longer an affine Springer fibre inside the affine Grassmannian Gr_G, but rather one has to consider such affine Springer fibres in the affine Flag variety $\mathrm{Fl}_G:=G((z))/\mathcal{I}$ (here $\mathcal{I}\subseteq G((z))$ denotes the *standard Iwahori subgroup*).

The inclusion $\mathcal{I}\subseteq G[[z]]$ induces a projection $\mathrm{Fl}_G\to\mathrm{Gr}_G$, and the $\mathbb{C}^\times$-action on Gr_G and its affine Springer fibres can be lifted to Fl_G. Then, the problem is solved by studying the relevant properties of upward flows in Fl_G under this action. Interestingly, if $w(E,\varphi,Q)$ is supported at D, the resulting upward flow turns out to be a section of h_G^{sp} (different from the natural, everywhere regular one obtained by adapting Hitchin's construction [77]), so in the parabolic setting we find an even simpler class of holomorphic Lagrangians that were not present in $\mathcal{M}(G)$.

Since these very stable upward flows are sections, they intersect each fibre at a single point. Therefore, the Fourier–Mukai transform realising the duality of branes indicates that the dual should simply be defined by a line bundle over $\mathcal{M}_{sp}(G^\vee,\alpha^\vee)$. We propose a construction in terms of a universal parabolic bundle

$$\mathbb{E}\to\mathcal{M}_{sp}(G^\vee,\alpha^\vee)\times C$$

as follows: being universal, it is endowed with *universal parabolic structures* $\mathbb{Q}_i$ which are reductions of structure group of $\mathbb{E}|_{M_{sp}(G^\vee,\alpha^\vee)\times\{c_i\}}$ to the Borel subgroup $B^\vee\subseteq G^\vee$. Denoting by $\mathbb{E}_i$ the corresponding $B^\vee$-bundle over $\mathcal{M}_{sp}(G^\vee,\alpha^\vee)$, and using the fact that the character lattice of $B^\vee$ equals the cocharacter lattice of T, we propose the following dual BBB-brane.

Conjecture (Conjecture 4.34). *Let (E,φ,Q) be a smooth fixed point of Borel type with twisted multiplicity divisor supported at D. Let $\mu_i(E,\varphi):=\mu_{c_i}(E,\varphi)$. The line bundle on $\mathcal{M}_{sp}(G^\vee,\alpha^\vee)$ mirror to the structure sheaf $\mathcal{O}_{W^+_{(E,\varphi,Q)}}$ on $\mathcal{M}_{sp}(G,\alpha)$ is given by*

$$\mathcal{L}_w:=\bigotimes_{c_i\in D}\mu_i^{-1}((\mathbb{E}_{\mathcal{M}_{sp}(G^\vee,\alpha^\vee)\times\{c_i\}})_{B^\vee})$$

up to tensoring by a fixed line bundle (independent of w) given by the chosen normalisation for $\mathbb{E}$.

In this formulation of the conjecture there are several implicit assumptions, such as the existence of $\mathbb{E}$ in the first place, or the fact that μ_i lifts to a cocharacter of the group. Besides these assumptions, checking such a conjecture requires enough understanding of the fibres of the Hitchin map and SYZ mirror symmetry, which so far seems to have only been developed for type A groups. All of these problems can be circumvented by working with $G=\mathrm{PGL}_n(\mathbb{C})$, which allows us to check the duality generically over $\mathcal{A}_{sp}(G)$ using the Fourier–Mukai transform on the smooth fibres.

Proposition (Proposition 4.40). *For $G=\mathrm{PGL}_n(\mathbb{C})$ and generic weights α in the interior of the standard alcove, the dual of the structure sheaf $\mathcal{O}_{W^+_{(E,\varphi,Q)}}$ for a fixed point (E,φ,Q) of Borel type with $w(E,\varphi,Q)$ supported at D equals the line bundle $\mathcal{L}_w$ proposed in the previous conjecture, over the generic locus of $\mathcal{A}_{sp}(\mathrm{PGL}_n(\mathbb{C}))$ corresponding to points with smooth spectral curves.*

This check is performed algebraically, leading to the natural question of whether this line bundle is hyperholomorphic with respect to the hyperkähler structure of the leaf. According to the physical predictions it should be (for it to define a BBB-brane), but it remains to be checked mathematically.

Structure and main results

This thesis is structured as follows. In Chapter 1, we provide the relevant background for moduli spaces of G-Higgs bundles and study their $\mathbb{C}^\times$-action. There is a first Section 1.1 devoted to introducing the necessary background in Lie theory that will be used throughout. Then, in Section 1.2 we recall the definition of the stability notions for Higgs bundles and explain the most relevant geometric properties of its moduli space $\mathcal{M}(G)$, as well as the Hitchin system h_G. In Section 1.3 we give an overview of different mirror symmetry statements, first as a general algebro-geometric phenomenon for Calabi–Yau varieties predicted by theoretical physics and then as a specific duality program for $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$. Section 1.4 recalls the description of fixed points of the $\mathbb{C}^\times$-action in terms of gradings of the Lie algebra and explains the main aspects of fixed points of Borel type (with generically regular Higgs field) and of Hermitian type. Section 1.5 explains the Białynicki-Birula upward flows in $\mathcal{M}(G)$ and defines very stable Higgs bundles, including Proposition 1.71 describing the upward flow of a given fixed point. In Section 1.6 we recall the definition and stability notions for parabolic Higgs bundles and the geometry of their moduli spaces, focusing on the strongly parabolic leaf $\mathcal{M}_{sp}(G,\alpha)$ and its integrable system h^{sp}_G. This section also contains an overview of the different mirror symmetry statements for $\mathcal{M}_{sp}(G,\alpha)$ that are present in the literature. Finally, in Section 1.7 we describe the fixed points of the $\mathbb{C}^\times$-action on $\mathcal{M}_{sp}(G,\alpha)$ in Proposition 1.92, a description which seems to be original in this generality.

Chapter 2 is devoted to the classification of very stable Higgs bundles of Borel type. First, we recall in Section 2.1 the definition and properties of the affine Grassmannian Gr_G and how it can be used to perform Hecke transformations of Higgs bundles. In Section 2.2 we showcase the induced $\mathbb{C}^\times$-action on Gr_G which is intertwined via the Hecke transformation process with the usual $\mathbb{C}^\times$-action on $\mathcal{M}(G)$ and we construct curves in Gr_G as $\mathbb{C}^\times$-orbits of this action. In Section 2.3 we prove the main classification theorem, Theorem 2.24, by studying how Hecke transformations relate different fixed points and when are the previous curves in Gr_G contained in the space of Hecke transformations of a given fixed point. In Section 2.4 we explain how the virtual equivariant multiplicity can be computed for a given fixed point of Borel type, and we verify in Proposition 2.32 that in the simplest very stable cases it recovers the Dynkin polynomial of the corresponding representation. We conclude in Section 2.5 by describing the finite cover given by the Hitchin map on these simplest very stable upward flows, which requires to consider families of affine Springer fibres parametrised by the Kostant section on $\mathfrak{g}$.

Then, in Chapter 3 we use the Toledo invariant and the Cayley correspondence to identify components without very stable Higgs bundles. In Section 3.1 we prove Theorem 3.5 which applies to fixed points with maximal Toledo invariant and non-split associated Hermitian form, while in Section 3.2 we prove Theorem 3.8 which shows how having a Toledo invariant with large absolute value is enough to imply wobbliness, ruling out many other fixed point components from having very stable Higgs bundles.

Chapter 4 contains the classification of very stable strongly parabolic G-Higgs bundles in the generically regular case. In Section 4.1 we recall the theory of affine flag varieties which generalises that of the affine Grassmannian and allows us to perform Hecke transformations in the presence of parabolic structures. In Section 4.2 we describe the $\mathbb{C}^\times$-action on the affine flag variety Fl_G which is compatible with the natural one on $\mathcal{M}_{sp}(G,\alpha)$ via Hecke transformations. We study two different kinds of curves in Fl_G (and its affine Springer fibres): one fully contained in a given fibre of the projection to Gr_G and other that moves along fibres. These two types are enough for the classification theorem, as they *generate* the basic moves that we can perform to connect different

upward flows. Then, in section 4.3 we prove the classification theorem, Theorem 4.33, by analysing when are the previous curves contained in the space of Hecke transformations of a given fixed point. We also construct the very stable upward flows for fixed points whose nontrivial multiplicities are supported at the parabolic punctures. Finally, in Section 4.4 we propose the dual line bundle under mirror symmetry for those very stable upward flows, and check the duality generically over the Hitchin base for $G = \mathrm{PGL}_n(\mathbb{C})$ using the spectral description of the fibres of $h^{sp}_{\mathrm{PGL}_n(\mathbb{C})}$ which we also recall.

We conclude with a last chapter which includes further directions to be pursued after the present work.

Chapter 1

Moduli spaces of Higgs bundles

1.1 Lie theory preliminaries and notation

First we review some Lie theory that will be needed throughout the next sections.

Let G be a connected, semisimple complex algebraic group with Lie algebra $\mathfrak{g}$. Fix a maximal torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T. This choice determines a Cartan subalgebra $\mathfrak{t} := \mathrm{Lie}(T)$ and a Borel subalgebra $\mathfrak{b} := \mathrm{Lie}(B)$ satisfying $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$. Denote by $\Delta := \Delta(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^*$ the corresponding root system and $\Pi = \{\beta_1, \dots, \beta_r\} \subseteq \Delta$ the choice of **simple roots** corresponding to $\mathfrak{b}$, where $r = \mathrm{rank}(\mathfrak{g})$. This determines a choice of **positive roots** $\Delta^+ \subseteq \Delta$. Consider also its dual basis $\{\omega_1^\vee, \dots, \omega_r^\vee\} \subseteq \mathfrak{t}$, which is the basis of **fundamental coweights**. We denote by $\delta \in \Delta$ the **highest root**.

The **root lattice**, which is the lattice in $\mathfrak{t}^*$ consisting of linear combinations with integer coefficients of the simple roots, is denoted by $Q_\Delta \subseteq \mathfrak{t}^*$. On the other hand, the **coweight lattice**, consisting of linear combinations with integer coefficients of the fundamental coweights, is denoted by $P_{\Delta^\vee} \subseteq \mathfrak{t}$. If $B(\cdot,\cdot)$ denotes the Killing form of $\mathfrak{g}$, and $B^*(\cdot,\cdot)$ its corresponding dual form on $\mathfrak{g}^*$, for any element $\beta \in \mathfrak{t}^*$ we define

$$\beta^\vee := 2\frac{B^*(\beta,\cdot)}{B^*(\beta,\beta)} \in \mathfrak{t},$$

and similarly from $\mathfrak{t}$ to $\mathfrak{t}^*$. We define the **coroot lattice** $Q_{\Delta^\vee} \subseteq \mathfrak{t}$ and the **weight lattice** $P_\Delta \subseteq \mathfrak{t}^*$ by applying this operation to the previous lattices, as well as the dual root system $\Delta^\vee$ and its positive root subset $\Delta^\vee_+$ by applying the operation to Δ and Δ^+, respectively. Similarly, we have sets of **fundamental weights** $\{\omega_1, \dots, \omega_r\} \subseteq \mathfrak{t}^*$ and **simple coroots** $\{\beta_1^\vee, \dots, \beta_r^\vee\} \subseteq \Delta^\vee$. The **Langlands dual group** $G^\vee$ is the semisimple group defined by the property of having $Q_{\Delta^\vee} \subseteq P_{\Delta^\vee}$ as it root lattice inside of its weight lattice, and $Q_\Delta \subseteq P_\Delta$ as its coroot lattice inside of its coweight lattice.

The coweights whose pairing with any positive root is non-negative are called **dominant**, defining the subset $P^+_{\Delta^\vee} \subseteq P_{\Delta^\vee}$. Similarly, if the pairing with any positive root is non-positive, they are called **antidominant** and denoted by $P^-_{\Delta^\vee} \subseteq P_{\Delta^\vee}$. Dually one defines notions of dominant and antidominant weights $P^+_\Delta, P^-_\Delta \subseteq P_\Delta$.

We also define the set of **characters** and **cocharacters** of T, which are the algebraic group homomorphisms

$$T \to \mathbb{G}_m,$$

and

$$\mathbb{G}_m \to T,$$

respectively. The groups formed by these are denoted by $X^*(T)$ and $X_*(T)$ respectively. Naturally, they induce linear morphisms $\mathfrak{t} \to \mathbb{C}$ and $\mathbb{C} \to \mathfrak{t}$, therefore they can be viewed in $\mathfrak{t}^*$ and $\mathfrak{t}$, respectively. We have inclusions

$$Q_\Delta \subseteq X^*(T) \subseteq P_\Delta$$

and

$$Q_{\Delta^\vee} \subseteq X_*(T) \subseteq P_{\Delta^\vee},$$

therefore dominant and antidominant characters and cocharacters can be considered as well, defining $X^*_+(T), X^*_-(T) \subseteq X^*(T)$, $X^+_*(T), X^-_*(T) \subseteq X_*(T)$.

A closed subgroup $P \subseteq G$ is called **parabolic** if the quotient G/P has the structure of a projective variety. Equivalently, P is parabolic if it contains a Borel subgroup of G. Such Borel subgroups are given by the conjugates of B under the action of G. If $B \subseteq P \subseteq G$, we say that P is a **standard parabolic** subgroup of G. Similarly, the Lie algebras $\mathfrak{p} \subseteq \mathfrak{g}$ corresponding to parabolic subgroups are called parabolic subalgebras and those containing $\mathfrak{b}$ are called standard.

Given a semisimple element $X \in \mathfrak{g}$, meaning that $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ diagonalises, and assuming that $\mathrm{ad}(X)$ has real eigenvalues, we can associate a parabolic subgroup

$$P_X := \{g \in G : \lim_{t \to -\infty} \exp(tX) g \exp(-tX) \text{ exists}\}$$

whose Lie algebra is the sum of the eigenspaces for non-negative eigenvalues of $\mathrm{ad}(X)$. If X is taken in $\mathfrak{t}^+_{\mathbb{R}} \subseteq \mathfrak{t}$, the subset consisting of linear combinations of the fundamental coweights with non-negative real coefficients, the resulting parabolic subgroup and subalgebra are standard. In order to uniquely parametrise the standard parabolic subgroups, it suffices to consider subsets $S \subseteq \{\omega_1^\vee, \dots, \omega_r^\vee\}$ and take X to be the sum of the elements in the chosen S. In other words, standard parabolic subgroups are given by subsets of the vertices in the Dynkin diagram (equivalently, by subsets of Π).

Recall that an element $e \in \mathfrak{g}$ is called **nilpotent** if the endomorphism $\mathrm{ad}(e) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, and an element $X \in \mathfrak{g}$ is called **regular** if $\dim C_{\mathfrak{g}}(X) = \operatorname{rank} \mathfrak{g}$, where $C_{\mathfrak{g}}(X)$ denotes the centraliser in $\mathfrak{g}$ of X. We will need the following result regarding regular nilpotent elements.

Lemma 1.1. *Let $e \in \mathfrak{g}$ be a regular nilpotent element and $\mathfrak{p} \subseteq \mathfrak{g}$ a parabolic subalgebra. There is a unique conjugate $\mathrm{Ad}_g(\mathfrak{p}) \subseteq \mathfrak{g}$, with $g \in G$, such that $e \in \mathrm{Ad}_g(\mathfrak{p})$.*

Proof. Note that the element g is uniquely defined up to right multiplication by an element of the parabolic subgroup $P \subseteq G$ with Lie algebra $\mathfrak{p}$, since P is self-normalising. The result is well known when $\mathfrak{p} = \mathfrak{b}$ (e.g. [33, Proposition 3.2.14]), but it remains true for any parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$. Assume without loss of generality that $\mathfrak{b} \subseteq \mathfrak{p}$. Take $gP \in G/P$ such that $e \in \mathrm{Ad}_g(\mathfrak{p})$. Then, there must be $g'B \in G/B$ such that $e \in \mathrm{Ad}_{g'}(\mathfrak{b}) \subseteq \mathrm{Ad}_g(\mathfrak{p})$: since all Borel subalgebras are conjugate, this follows from the fact that any nilpotent element in $\mathrm{Ad}_g(\mathfrak{p})$ can be included in a Borel subalgebra, see e.g. [78, Lemma in Section 30.3]. We already know that this $g'B \in G/B$ is unique (independently of the chosen g, only using $e \in \mathrm{Ad}_{g'}(\mathfrak{b})$), and the inclusion $\mathrm{Ad}_{g'}(\mathfrak{b}) \subseteq \mathrm{Ad}_g(\mathfrak{p})$ implies that $gP = g'P$, showing the uniqueness as desired. □

Let $W = N_G(T)/T$ denote the **Weyl group** of $\mathfrak{g}$, which is a finite reflection group acting on T and on $\mathfrak{t}$ as well as $\mathfrak{t}^*$. It is generated by the set $S_W := \{s_1 := s_{\beta_1}, \dots, s_r := s_{\beta_r}\}$ of reflections about the simple root hyperplanes $\{\beta_j = 0\} \subseteq \mathfrak{t}$.

Similarly, we have the **affine Weyl group**

$$W_{\text{aff}} := Q_{\Delta^\vee} \rtimes W$$

which is another (non-finite) reflection group acting on $\mathfrak{t}$: the coroot lattice $Q_{\Delta^\vee}$ acts by translations, and we adopt the convention that $\mu \in Q_{\Delta^\vee}$ maps $X \in \mathfrak{t}$ to $X - \mu$; this convention shall become clearer in Section 4.1 when this group is regarded inside of the loop group of G. It is generated

by the set $S_{W_{\mathrm{aff}}} = S_W \cup \{s_0 := s_{\delta,1}\}$ where $s_{\delta,1}$ is the reflection by the affine root hyperplane $\{\delta = 1\} \subseteq \mathfrak{t}$ (recall that $\delta \in \Delta$ denotes the highest root).

We have two slightly larger groups that will be crucial when studying parabolic Higgs bundle moduli spaces in Chapter 4. The first one is the **Iwahori–Weyl group**

$$\widetilde{W}_G := X_*(T) \rtimes W$$

and the second one is the **extended affine Weyl group**

$$\widetilde{W} := P_{\Delta^\vee} \rtimes W.$$

The inclusions of lattices described above imply inclusions

$$W_{\mathrm{aff}} \subseteq \widetilde{W}_G \subseteq \widetilde{W}.$$

The last two groups are not reflection groups in general: the quotient

$$\widetilde{W}/W_{\mathrm{aff}} \simeq P_{\Delta^\vee}/Q_{\Delta^\vee}$$

is finite and elements in the nontrivial classes cannot be expressed as a product of root hyperplane reflections. A set of representatives of the quotient is that of **dominant minuscule coweights**, which are the elements $\mu \in P_{\Delta^\vee}$ such that $\beta(\mu) \in \{0, 1\}$ for every $\beta \in \Delta^+$. The quotient $\widetilde{W}_G/W_{\mathrm{aff}} \subseteq \widetilde{W}/W_{\mathrm{aff}}$ can be identified with the set of dominant minuscule coweights contained in $X_*(T)$. Dominant minuscule coweights are always fundamental, see e.g. Table 2.1 for their classification on each simple Lie algebra. In general, a **minuscule** coweight is any $\mu \in P_{\Delta^\vee}$ in the Weyl group orbit of a dominant minuscule coweight, i.e. any $\mu \in P_{\Delta^\vee}$ such that $\beta(\mu) \in \{\pm 1, 0\}$ for every $\beta \in \Delta$.

All these Weyl groups fit (up to finite index in the last two cases) within the theory of Coxeter groups. In what follows, we recall the necessary aspects of this theory for later use, a good reference is [21]. We denote by $(\widehat{W}, \widehat{S})$ a Coxeter system, for our purposes this is either (W, S_W) or $(W_{\mathrm{aff}}, S_{W_{\mathrm{aff}}})$. Similarly, we denote by $(\widehat{W}', \widehat{S})$ a quasi-Coxeter system, meaning that $\widehat{W}'$ is a group containing some Coxeter group $\widehat{W}$ (which is part of a Coxeter system $(\widehat{W}, \widehat{S})$) as a normal subgroup such that the quotient $\widehat{W}'/\widehat{W}$ is finite abelian. For our purposes this is either $(\widetilde{W}_G, S_{W_{\mathrm{aff}}})$ or $(\widetilde{W}, S_{W_{\mathrm{aff}}})$, in both cases containing W_{aff}. We have a natural splitting $\widehat{W}' = \widehat{W} \rtimes \Omega$, where $\Omega \simeq \widehat{W}'/\widehat{W}$ is defined to be the normaliser of $\widehat{S}$.

Definition 1.2 *The **length** of an element $w' = w \cdot \omega \in \widehat{W}' = \widehat{W} \rtimes \Omega$ is the minimal number $k \in \mathbb{Z}_{\geqslant 0}$ such that there is an expression*

$$w = s_{i_1} \cdot \cdots \cdot s_{i_k}$$

for $s_{i_j} \in \widehat{S}$. We denote it by $l(w')$.

Definition 1.3 *The **Bruhat order** in $\widehat{W}' = \widehat{W} \rtimes \Omega$ is defined by the following rule: for $w \cdot \omega, w' \cdot \omega' \in \widehat{W}'$, we have $w \cdot \omega \leqslant w' \cdot \omega'$ if and only if both of these hold:*

- *The length zero parts agree: $\omega = \omega'$.*
- *Given an expression $w' = s_{i_1} \cdot \cdots \cdot s_{i_k}$ where $s_{i_j} \in \widehat{S}$, there exists a subexpression, obtained by omitting some entries, whose product equals w.*

Some basic properties that we use frequently are that length is order-preserving and that the Bruhat order is generated by relations of the form $w \leqslant wt$ for $w \in \widehat{W}$, $t = usu^{-1}$ with $u \in \widehat{W}$, $s \in \widehat{S}$ and $l(w) \leqslant l(wt)$. Moreover, inverting elements preserves length.

Definition 1.4 *Given a subset $J \subseteq \widehat{S}$, we define the **parabolic subgroup** $\widehat{W}_J \subseteq \widehat{W}$ as the subgroup generated by J. The pair $(\widehat{W}_J, J)$ is itself a Coxeter system. The quotient $\widehat{W}'/\widehat{W}_J$ has a natural set of representatives given by those with minimal length:*

$$(\widehat{W}')^J := \{w \in \widehat{W}' : l(ws) > l(w) \text{ for all } s \in J\}.$$

*This set is called the **left parabolic quotient** for J. Analogously, the quotient $\widehat{W}_J\backslash\widehat{W}'$ has minimal length representatives:*

$${}^J\widehat{W}' := \{w \in \widehat{W}' : l(sw) > l(w) \text{ for all } s \in J\} = \left((\widehat{W}')^J\right)^{-1}.$$

*This set is called **right parabolic quotient** for J. Similarly, for subsets $J, J' \subseteq \widehat{S}$ the double quotient $\widehat{W}_{J'}\backslash\widehat{W}'/\widehat{W}_J$ has a natural set of representatives:*

$${}_{J'}\left(\widehat{W}'\right)^J = \left\{w \in \widehat{W}' : l(w) = \max_{w_1 \in \widehat{W}_{J'}} \min_{w_2 \in \widehat{W}_J} l(w_1 w w_2)\right\}.$$

*This set is called the **double parabolic quotient** for J, J'.*

The Bruhat order and length restrict to parabolic quotients and double parabolic quotients in such a way that the quotient morphisms are order preserving. Note that for double quotients it is also standard to take ${}^{J'}(\widehat{W}')^J$ the set of minimum length representatives (i.e. not mixing minimum and maximum), but we will see later in Section 4.1 that geometrically the previous choice is cleaner.

Remark 1.5 *In the case of the finite Weyl group (W, S_W), parabolic quotients have a unique longest element which is of special importance to us. It suffices to recall that the length of an element in W can be regarded as the number of positive roots which are sent to negative roots. Since elements $w \in W^J$ are characterised by $l(ws) > l(w)$ for $s \in J$ or, equivalently, by sending the simple roots $\Pi_J \subseteq \Pi$ whose associated reflections are in J to positive roots, the unique longest element is the one sending every positive root outside the span of Π_J to a negative root. Inverting, the same is true for ${}^JW$ by considering w^{-1}.*

Remark 1.6 *Given an element $\mu \in \mathfrak{t}$, we often associate parabolic subgroups and quotients by taking $J_\mu \subseteq \widehat{S}$ to be the stabiliser of μ.*

We conclude this section by identifying precisely the length zero elements (i.e. the set Ω) for the extended affine Weyl group $\widetilde{W}$. It turns out that this does not simply agree with the set of minuscule representatives we gave earlier. In our convention (where $P_{\Delta^\vee}$ translates with a negative sign), we have the length formula:

$$l(\mu \cdot u) = \sum_{\beta, u^{-1}(\beta) \in \Delta^+} |\mu(\beta)| + \sum_{\beta, -u^{-1}(\beta) \in \Delta^+} |\mu(\beta) + 1|.$$

This was given in [81, Proposition 1.23], noting that they provide the formula in the case that $P_{\Delta^\vee}$ translates by adding.

Proposition 1.7. *Let $\mu \cdot u \in \widetilde{W} = P_{\Delta^\vee} \rtimes W$. Then, $l(\mu \cdot u) = 0$ if and only if*

- *μ is antidominant minuscule, i.e. $\mu(\beta) \in \{0, -1\}$ for all $\beta \in \Delta^+$.*
- *u is the longest element in the parabolic quotient (of W)*

$${}^{J_\mu}W = \{w \in W : w^{-1}(\beta) \in \Delta^+ \text{ for all } \beta \in \Pi \text{ such that } \mu(\beta) = 0\}.$$

Proof. It suffices to use the length formula given immediately above. For the length to be zero, it is clear that $\mu(\beta) \in \{0, -1\}$ for positive roots $\beta \in \Delta^+$. Moreover, we need that $\mu(\beta) = 0$ if and only if $u^{-1}(\beta) \in \Delta^+$. This characterises u as the longest element of ${}^{J_\mu}W$. Both conditions are clearly sufficient. □

1.2 The moduli space of G-Higgs bundles

Now we define the moduli space of G-Higgs bundles, which is the main object of study in this thesis. We denote the base curve by C, which is a smooth projective complex curve of genus $g \geqslant 2$ with canonical line bundle K_C.

Definition 1.8 *A G-**Higgs bundle** over C is a pair (E, φ) where E is a principal G-bundle over C and $\varphi \in H^0(C, E(\mathfrak{g}) \otimes K_C)$.*

Here $E(\mathfrak{g}) := E \times_{\mathrm{Ad}} \mathfrak{g}$ is the adjoint bundle, that is, the vector bundle associated to the adjoint representation of G. The section φ is usually called *Higgs field*.

Example 1.9 *The definition of a G-Higgs bundle can be restated in terms of vector bundles over C with extra structure whenever G is a classical group, by using the natural embedding of G into $\mathrm{GL}_N(\mathbb{C})$ for some N. For example, if we take $G = \mathrm{SL}_n(\mathbb{C})$ then a principal G-bundle E can be identified via the standard representation as a holomorphic vector bundle of rank n with $\det(E) \simeq \mathcal{O}_C$. Denoting such bundle by E as well, the Higgs field is given by a traceless vector bundle morphism $\varphi : E \to E \otimes K_C$, since $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ is given by the traceless endomorphisms of $\mathbb{C}^n$. Another example is $G = \mathrm{Sp}_{2n}(\mathbb{C})$, which corresponds to rank $2n$ vector bundles E over C endowed with a symplectic structure $\Omega : E \otimes E \to \mathcal{O}_C$ such that the Higgs field $\varphi : E \to E \otimes K_C$ verifies $\Omega(\varphi v, w) + \Omega(v, \varphi w) = 0$.*

In order to introduce the moduli space, which parametrises isomorphism classes of polystable G-Higgs bundles, we need to recall the suitable notions of stability [52]. We start with the following definition:

Definition 1.10 *Let $\hat{G} \leqslant G$ be a closed subgroup, and E a principal G-bundle. A **reduction of structure group** of E to $\hat{G}$ is a section $\sigma \in H^0(C, E(G/\hat{G}))$, where $E(G/\hat{G}) = E \times_G G/\hat{G}$.*

Given such a section $\sigma : X \to E(G/\hat{G})$, it is possible to pull back the $\hat{G}$-bundle E on $E(G/\hat{G})$ which results in $E_\sigma := \sigma^* E$, a $\hat{G}$-bundle on X. This motivates the term *reduction*. There is a canonical isomorphism $E_\sigma \times_{\hat{G}} G \simeq E$ and the map $E_\sigma = \sigma^* E \to E$ induced by the pullback gives a subvariety $E_\sigma \subseteq E$.

The stability conditions for G-Higgs bundles are defined via reductions to parabolic subgroups of G. Let $P \subseteq G$ be a standard parabolic subgroup (recall that this means P is a closed subgroup containing B). Let $L \subseteq P$ be its standard Levi subgroup, by which we mean the Levi subgroup that contains T. Choose a dominant character $\chi \in X^*_+(P)$, i.e. a character $\chi : P \to \mathbb{G}_m$ such that $\chi|_T$ is dominant.

Given a principal G-bundle E and a reduction $\sigma \in H^0(E(G/P))$ we may define a notion of **degree** of σ with respect to the dominant character χ, given by

$$\deg(E, \sigma, \chi) := \deg \chi(E_\sigma),$$

where we use that $\chi(E_\sigma)$ is a $\mathbb{C}^\times$-bundle, i.e. a line bundle.

Remark 1.11 *The degree of E with respect to a parabolic reduction and character can also be defined using analytic techniques via Chern–Weil theory. This is accomplished by reducing E_σ further to the maximal compact subgroup of L, yielding $E_{\sigma'}$, choosing a connection A on $E_{\sigma'}$ with curvature F_A and setting*

$$\deg(E, \sigma, \chi) := \frac{i}{2\pi} \int_C \chi(F_A).$$

We can now define stability (see [52]).

Definition 1.12 *A G-Higgs bundle (E, φ) is:*

- ***semistable**, if for any standard parabolic subgroup $P \subseteq G$, dominant character $\chi \in X_*^+(P)$ and reduction $\sigma \in H^0(C, E(G/P))$ such that $\varphi \in H^0(C, E_\sigma(\mathfrak{p}) \otimes K_C) \subseteq H^0(C, E(\mathfrak{g}) \otimes K_C)$, we have* $\deg(E, \sigma, \chi) \leqslant 0$.
- ***stable**, if for any proper standard parabolic subgroup $P \subsetneq G$, nontrivial dominant character $\chi \in X_*^+(P) \setminus \{0\}$ and reduction $\sigma \in H^0(C, E(G/P))$ such that $\varphi \in H^0(C, E_\sigma(\mathfrak{p}) \otimes K_C) \subseteq H^0(C, E(\mathfrak{g}) \otimes K_C)$, we have* $\deg(E, \sigma, \chi) < 0$.
- ***polystable**, if it is semistable and, for any standard parabolic subgroup $P \subseteq G$, dominant character $\chi \in X_*^+(P)$ and reduction $\sigma \in H^0(C, E(G/P))$ such that $\varphi \in H^0(C, E_\sigma(\mathfrak{p}) \otimes K_C) \subseteq H^0(C, E(\mathfrak{g}) \otimes K_C)$ and $\deg(E, \sigma, \chi) = 0$, there exists a reduction $\sigma' \in H^0(E_\sigma(P/L))$ of E_σ to L such that $\varphi \in H^0(C, E_{\sigma'}(\mathfrak{l}) \otimes K_C) \subseteq H^0(C, E_\sigma(\mathfrak{p}) \otimes K_C)$.*

Note that any stable G-Higgs bundle is automatically semistable, and (by definition) any polystable Higgs bundle is also semistable.

There exists a **moduli space of isomorphism classes of polystable G-Higgs bundles over C** which is a complex, quasiprojective algebraic variety of dimension $2(g-1)\dim(G)$ which can, in general, have singularities. It was introduced by Hitchin [75] and constructed in this generality by Simpson [115, 117]. We denote it by $\mathcal{M}(G)$.

Remark 1.13 *A more general construction using Geometric Invariant Theory can be found in the book of Schmitt [111]. This construction includes stability and moduli spaces for (G, V)-*Higgs pairs*, where V is any representation of G (not necessarily the adjoint one). In Section 1.4 examples of such objects will appear to describe the fixed point locus of the $\mathbb{C}^\times$-action on $\mathcal{M}(G)$. However, we will not need to state these notions of stability explicitly.*

Similarly, the case where G is not connected has been studied in [51]. Nevertheless, in the present work all the structure groups appearing in questions related to stability will be connected.

The stability conditions for G-Higgs bundles arise naturally when considering solutions to **Hitchin's equation**, a gauge-theoretic Hermite–Einstein-like equation regarding the existence of certain metric on the smooth bundle underlying E. We now review this, as it is the starting point for an extremely rich geometric structure in $\mathcal{M}(G)$ which plays a role in mirror symmetry. First, we recall the notion of a Hermitian metric on a principal G-bundle. Fix $K \leqslant G$ a maximal compact subgroup and let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the conjugate-linear involution such that $\mathfrak{g}^\tau = \mathfrak{k} := \mathrm{Lie}(K)$. Since K is a real Lie group, the associated bundle $E(G/K)$ is only smooth, not algebraic.

Definition 1.14 *A **Hermitian metric** on E is a smooth section $h \in H^0(E(G/K))$. In other words, it is a smooth reduction of structure group of E to K.*

As usual, we denote by E_h the resulting (smooth) principal K-bundle. Recall that a Hermitian metric h on E pinpoints a distinguished connection A_h on E_h, the **Chern–Singer connection** [118]. It is characterised by a compatibility condition: the $(0,1)$-part of the induced connection A_h^G on E coincides with the Dolbeault operator given by the complex structure of E. We let $F_h \in \Omega^{1,1}(E_h(\mathfrak{k}))$ be its curvature.

Moreover, the involution τ induces a conjugate-linear involution $\tau_h : E(\mathfrak{g}) \to E(\mathfrak{g})$, since τ fixes $\mathfrak{k}$ and $E_h(\mathfrak{g}) \simeq E(\mathfrak{g})$. This induces involutions τ on sections, as well as on differential forms by also conjugating the form-part. In particular, regarding the Higgs field as a smooth form $\varphi \in \Omega^{1,0}(E(\mathfrak{g}))$ gives $\tau_h(\varphi) \in \Omega^{0,1}(E(\mathfrak{g}))$.

With all the ingredients in place, we have the following result, originally due to Hitchin and Simpson.

Theorem 1.15 (Hitchin–Kobayashi correspondence [75, 115]). *A G-Higgs bundle (E, φ) is polystable if and only if it admits a Hermitian metric $h \in H^0(E(G/K))$ such that*

$$F_h + [\varphi, -\tau_h(\varphi)] = 0.$$

There exist more general versions of the theorem regarding (G, V)-Higgs pairs (as in Remark 1.13), for example [52]. These play an important role in the generalisation of the theory of Higgs bundles to real form $G_{\mathbb{R}} \subseteq G$ structure groups.

The Hitchin–Kobayashi correspondence is the first part of a celebrated result relating Higgs bundles with representations of the fundamental group $\pi_1(C)$ into G. We now recall the second part. It suffices to notice that the term $F_h + [\varphi, -\tau_h(\varphi)] \in \Omega^{1,1}(E_h(\mathfrak{k}))$ can be regarded as the curvature of the connection on (the underlying smooth bundle) E defined by the covariant derivative

$$D_h := d_{A_h} + \varphi - \tau_h(\varphi)$$

on the adjoint bundle.

Therefore, Hitchin's equation is equivalent to the flatness of D_h. Its holonomy defines a representation

$$\rho : \pi_1(C) \to G$$

which is moreover **completely reducible**, that is, its composition with the adjoint representation, which is a $\pi_1(C)$ representation on the vector space $\mathfrak{g}$, splits as a direct sum of irreducible representations.

Conversely, given such a completely reducible representation ρ, one can easily construct a smooth G-bundle E_ρ with a flat connection D_ρ. Then, an important result of Corlette [35] and Donaldson [40] shows that the complete reducibility guarantees the existence of a metric $h \in H^0(E(G/K))$ which is **harmonic**. This means the following: after decomposing D_ρ into its projection to $\mathfrak{k}$, which gives a connection d_A on E_h, and considering the remaining part $\psi \in \Omega^1(E_h(i\mathfrak{k}))$, one may set φ to be the $(1,0)$ part of ψ. One also gets a complex structure on E_ρ defined by the $(0,1)$ part of d_A, giving a holomorphic G-bundle E on C. Harmonicity means that (E, φ) defines a G-Higgs bundle over C (in other words, that φ is a holomorphic section of E), as well as Hitchin's equation

$$F_h + [\varphi, -\tau_h(\varphi)] = 0$$

being satisfied. In other words, by Theorem 1.15 we recover a polystable G-Higgs bundle.

We set $\mathcal{R}(G)$ to be the **character variety** of C, that is, the moduli space of completely reducible representations $\pi_1(C) \to G$ up to conjugation. Algebraically, this is nothing but the affine GIT quotient

$$\mathcal{R}(G) := \mathrm{Hom}(\pi_1(C), G) /\!\!/ G,$$

since the polystable representatives are the completely reducible representations. We can then summarise the above discussion as follows.

Theorem 1.16 (Non-abelian Hodge correspondence [35, 40, 75, 115]). *There is a homeomorphism*

$$\mathcal{M}(G) \simeq \mathcal{R}(G).$$

It is clear from the previous discussion that the correspondence is not algebraic in nature, as we had to resort to strictly differential-geometric techniques on the underlying smooth objects.

Moreover, there is no hope for such an algebraic isomorphism since $\mathcal{R}(G)$ is an affine variety whereas $\mathcal{M}(G)$ is quasiprojective and not affine: it contains positive-dimensional subvarieties which are projective, such as $\mathcal{N}(G) \subseteq \mathcal{M}(G)$ the moduli space of isomorphism classes of polystable G-bundles over C, via the embedding $E \mapsto (E, 0)$.

Remarkably, this means that we have two different complex structures on the underlying smooth manifold for the smooth part $\mathcal{M}^s(G)$: the original one coming from the construction as a moduli space of Higgs bundles, which we denote by I, and the induced one from $\mathcal{R}(G)$ via the non-abelian Hodge correspondence, which we denote by J. We can get a third one simply by setting $K := IJ$, showcasing a full sphere of complex structures in the directions of I, J, K.

Moreover, for each of these complex structures there is a compatible Kähler form $\omega_I, \omega_J, \omega_K \in \Omega^2(\mathcal{M}^s(G))$. This gives the smooth locus $\mathcal{M}^s(G)$ the structure of a **hyperkähler manifold**, and makes $\mathcal{M}(G)$ into a hyperkähler variety. This gives a **complex symplectic form**

$$\omega := \omega_J + i\omega_K,$$

which is holomorphic with respect to I. Algebraically we obtain a complex symplectic form $\omega \in H^0(\Omega^2_{\mathcal{M}^s(G)})$.

An important feature of the moduli space $\mathcal{M}(G)$ is the existence of an algebraic completely integrable system on it [76] with respect to the complex symplectic form ω. It is defined by using that the ring of invariant polynomial functions $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, \dots, p_r]$ is a polynomial ring on $r = \operatorname{rank} \mathfrak{g}$ generators [32, Theorem A].

Definition 1.17 *The **Hitchin map** is defined by*

$$\begin{aligned} h_G : \mathcal{M}(G) &\to \mathcal{A}(G) := \bigoplus_{i=1}^{r} H^0(C, K_C^{\deg p_i}) \\ (E, \varphi) &\mapsto (p_1(\varphi), \dots, p_r(\varphi)). \end{aligned}$$

The Hitchin map is a proper map of algebraic varieties, a fact which will be of great importance later.

Let us now recall (see e.g. [19, Theorem 2.3], [52, Section 3.3]) the algebraic deformation theory for G-Higgs bundles, which allows to identify the tangent space of $\mathcal{M}(G)$ at its smooth points.

Definition 1.18 *Let (E, φ) be a G-Higgs bundle. Its **deformation complex** is the complex of sheaves defined by*

$$C^\bullet(E, \varphi) : E(\mathfrak{g}) \xrightarrow{[\varphi, -]} E(\mathfrak{g}) \otimes K_C.$$

Proposition 1.19 ([19, Theorem 2.3]). *The space of infinitesimal deformations of a G-Higgs bundle (E, φ) is naturally isomorphic to $\mathbb{H}^1(C^\bullet(E, \varphi))$.*

The main idea for the proof, which we shall use later, is that under a finite covering $C = \bigcup_{i \in I} U_i$ by open affine subvarieties, a hypercohomology class can be seen as a Čech cocycle $\{(s_{ij}, t_i)\}_{i,j \in I}$ such that $s_{ij} \in H^0(U_i \cap U_j, E(\mathfrak{g}))$ are used to glue the trivial deformation $E|_{U_i} \times \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$ by using the transition functions $1_G + \varepsilon s_{ij}$ on the intersections $U_i \cap U_j$, and $t_i \in H^0(U_i, E(\mathfrak{g}) \otimes K_C)$ are used to deform the Higgs field by $\varphi'|_{U_i} := \varphi|_{U_i} + \varepsilon t_i$.

In order to pinpoint the smooth locus of $\mathcal{M}(G)$, we need to recall the notion of *simple* G-Higgs bundle.

Definition 1.20 *A G-Higgs bundle (E, φ) is **simple** if $\operatorname{Aut}(E, \varphi) = Z(G)$.*

Then, we have [52, Proposition 3.18] the following smoothness result.

Proposition 1.21. *If the G-Higgs bundle (E,φ) is stable and simple, it defines a smooth point in $\mathcal{M}(G)$.*

We conclude by remarking that the complex symplectic form ω can be constructed using the previous deformation theory. Indeed, by Serre duality we have

$$\mathbb{H}^1(C^\bullet(E,\varphi))^* \simeq \mathbb{H}^1(C^\bullet(E,\varphi)^* \otimes K_C) \simeq \mathbb{H}^1(C^\bullet(E,\varphi)),$$

since $E(\mathfrak{g})$ is self dual as can be seen by using the Killing form B. This pairing on the tangent space to $\mathcal{M}(G)$ induces the symplectic form ω.

1.3 Mirror symmetry phenomena of $\mathcal{M}(G)$

The term **mirror symmetry** refers to a kind of duality that first arose in theoretical physics, more concretely in the study of certain supersymmetric string theories. While these physical theories are not yet rigorously defined from a mathematical standpoint, thus there is no precise statement of mirror symmetry at this level, the underlying physical intuition has allowed to extract several concrete mathematical dualities, some still conjectural, which collectively carry the name of *mirror symmetry* and should follow as a consequence of these physical dualities. The first such example [31] was the proposal of a relationship, guided by physical intuition, between the number of rational curves on certain quintic Calabi–Yau manifold in terms of a computation of periods on a different, *mirror* Calabi–Yau manifold.

The goal of the present section is to state some of the main mirror symmetry statements in mathematics and recall how $\mathcal{M}(G)$ fits into these statements, a program started by Hausel and Thaddeus [73] (see also [68] for a recent survey). For the general mirror symmetry statements we mainly follow [4, 36].

First let us overview some of the mathematical aspects of supersymmetric string theories that will be relevant to us. These theories are concerned with the propagation of strings (one dimensional manifolds) in spacetime. Such propagation can be modelled by a map $\Sigma \to \mathbb{R}^{1,3} \times V$ where Σ is a Riemann surface (called *worldsheet*) which models the propagation in time of the string, while the *target space* $\mathbb{R}^{1,3} \times V$ represents spacetime. Notably, there is a compact, six-dimensional factor V which seems to be required for the mathematical consistency of the theories. Physicists think about these maps in terms of field theory, where they become fields on Σ taking values in the target space. Altogether, this study of fields with values in a non-euclidean space is referred to as a *non-linear sigma model* in physics.

The physical requirements for the theories translate into mathematical constraints on V. In physical jargon, the relevant requirements are that the result should be a $N = (2,2)$ *supersymmetric conformal field theory* (SCFT). The first constraint, supersymmetry, refers to the existence of several *supercharge* operators (two of holomorphic nature and two of antiholomorphic nature) preserving the theory. The relevant mathematical consequence for us is that, for these to exist, V must have a Kähler metric, so we impose that V is a complex variety with a Kähler class $\omega \in H^2(V,\mathbb{R})$. The second constraint, conformality, refers to the theory being invariant under conformal transformations of Σ. Mathematically, this implies that the class ω should admit a Ricci-flat representative. By Yau's theorem this is equivalent to requiring $c_1(V) = 0$. If V is simply connected, this is equivalent to the canonical bundle of V being trivial. This means that we are considering V a compact **Calabi–Yau** threefold together with a Kähler class ω. More generally we can take V to be any smooth projective variety over $\mathbb{C}$ with a Kähler class $\omega \in H^2(V,\mathbb{R})$ and with $K_V \simeq \mathcal{O}_V$.

The construction of these theories starting from (V,ω) depend on some specific sign choices which suggest the existence of a different pair $(V^\vee,\omega^\vee)$ with the same characteristics, usually termed the **mirror** (we say that (V,ω) and $(V^\vee,\omega^\vee)$ form a **mirror pair**), in such a way that the resulting physics constructed from either (V,ω) or $(V^\vee,\omega^\vee)$ (with the above-stated choices being made differently) are isomorphic. As stated above, this is not a rigorous claim since the process of getting a physical theory from the mathematical data of (V,ω) is not well defined. However, if true, it would imply some rigorous statements.

The first one is **topological mirror symmetry**, which implies the agreement of certain topological invariants of V and $V^\vee$. Indeed, the supersymmetric aspects of the theory imply the existence of two distinguished operators (generators, called *left/right moving R-symmetries*, of a copy of $\mathfrak{u}(1)\times\mathfrak{u}(1)$ inside the Lie algebra of the symmetry group of the theories, called *superconformal algebra*) whose eigenspaces with eigenvalue (p,q) for $p,q\geqslant 0$ in the space of states are given by $H^q(V,\wedge^p T_V)$ and for $(-p,q)$ they are given by $H^q(V,\Omega_V^p)$.

One of the choices for the theory is the sign of these operators: changing the sign of the first operator should not matter from a physical standpoint. This implies that there should be isomorphisms

$$H^q(V,\wedge^p T_V)\simeq H^q(V^\vee,\Omega^p_{V^\vee})$$
$$H^q(V^\vee,\wedge^p T_{V^\vee})\simeq H^q(V^\vee,\Omega^p_{V^\vee}).$$

Note that this already showcases that $V^\vee$ could be different to V since usually these spaces are of different dimensions for a given V. Using the triviality of K_V, we can identify $H^q(V,\wedge^p T_V)\simeq H^q(V,\Omega_V^{n-p})$ where $n=\dim_{\mathbb{C}}(V)$. This implies a relation between the **Hodge numbers** $h_V^{p,q}:=\dim H^q(V,\Omega_V^p)$.

Conjecture 1.22 (Topological mirror symmetry) *A mirror pair (V,ω), $(V^\vee,\omega^\vee)$ satisfies $h_V^{p,q}=h_{V^\vee}^{n-p,q}$.*

This means that the Hodge diamond for V is a mirror image of that for $V^\vee$, motivating the name mirror symmetry. One commonly heard slogan for mirror symmetry is that it interchanges complex geometry one one Calabi–Yau manifold with symplectic geometry on its mirror. This is already present here: $h_V^{1,1}=\dim H^{1,1}(V)$ parametrises the (complex) deformations of the Kähler class ω, where a cone inside of $H^{1,1}(V)$ results in Kähler deformations. This agrees with $h_{V^\vee}^{n-1,1}=\dim H^1(V^\vee,\Omega_{V^\vee}^{n-1})\simeq H^1(V^\vee,T_{V^\vee})$ (again, for the last step we use that $K_{V^\vee}$ is trivial), which is the space of deformations of the complex structure of $V^\vee$.

Of course, topological mirror symmetry showcases a consequence of having a mirror pair: mathematically this means that for every (V,ω) there is a mirror $(V^\vee,\omega^\vee)$ with that topological constraint. This is not very enlightening as it does not explain how to construct $(V^\vee,\omega^\vee)$ nor does it imply a deeper relation between the members of the pair. There have been several proposals to clarify this regard, the most relevant to us is that of Strominger–Yau–Zaslow [120], referred to as **SYZ mirror symmetry**.

The underlying idea of their proposal is to exploit the physical notion of *T-duality*, which posits an equivalence of the physical theory constructed with a d-dimensional torus as a target space with that having a *dual torus* as its target (where dual here means inverting the metric on the torus). The proposal of SYZ mirror symmetry is that both V and $V^\vee$ should admit torus fibrations over the same base B, and mirror symmetry should come from applying this T-duality to each of the fibres.

Conjecture 1.23 (SYZ mirror symmetry) *A mirror pair formed by (V,ω) and $(V^\vee,\omega^\vee)$ admits continuous surjective morphisms $f:V\to B$, $f^\vee:V^\vee\to B$ such that there is a dense open*

subset $B_{\text{reg}} \subseteq B$ over which $f, f^\vee$ are fibrations by nonsingular, special Lagrangian tori, and for $b \in B_{reg}$ we have that $f^{-1}(b)$ and $(f^\vee)^{-1}(b)$ are dual tori.

There are two clarifications in order: first, *dual tori* should mean that they correspond to each other under T-duality, at least in some limit. A more precise statement would be that $(f^\vee)^{-1}(b)$ is given by the space of flat $U(1)$-connections $\text{Hom}(H_1(f^{-1}(b), \mathbb{Z}), U(1))$ on the other. In our setting, for $\mathcal{M}(G)$ we will see that these tori have the structure of a complex algebraic variety so we can more simply say that they are dual abelian varieties. Second, we need to explain what we mean by *special Lagrangian* and why is it imposed: for this, we need to take a detour and explain the notion of **branes**.

Physically, branes (more precisely *Dirichlet branes* or *D-branes*) represent boundary conditions for the theory. Indeed, when modelling an open string, the worldsheet Σ has boundaries: we restrict that these must lie in a brane contained in V.

Definition 1.24 *A **brane** in V is a triple (L, E, ∇) where $L \subseteq V$ is a submanifold, E is a vector bundle on L and ∇ is a connection on E.*

The submanifold $L \subseteq V$ represents the locus where the boundary of Σ is constrained, whereas E and ∇ are physical, gauge–theoretic requirements: roughly speaking, E allows for local deformations of the theory and ∇ accounts for the weight of the field in the functional integral of the theory, which will be related to the holonomy induced by the boundary of Σ with respect to this connection.

We now turn our attention to which branes are stable, energy-minimising (in physics, known as *BPS states*). Again, the answer will differ depending on the choices made when constructing the theory, and mirror symmetry is expected to interchange the two sets. One of the two types is easy to describe.

Definition 1.25 *A brane (L, E, ∇) in V is said to be of **type B** if $L \subseteq V$ is a complex subvariety, both E and ∇ are holomorphic, and the curvature of ∇ is constant central.*

On the other hand, it is unclear what the complete definition of the other type should be. We can however describe a family of branes which minimise the energy.

Definition 1.26 *A brane (L, E, ∇) in V is **special Lagrangian** if $L \subseteq V$ is Lagrangian with respect to ω, the connection ∇ is flat and the imaginary part of the nonvanishing section Ω of K_V vanishes along L. Special Lagrangian branes are of **type A**.*

Other branes of type A have been identified [82] and they have coisotropic supports L.

Conjecture 1.27 *The space of A-branes of the theory constructed from (V, ω) is isomorphic to the space of B-branes of the dual theory constructed from $(V^\vee, \omega^\vee)$.*

With these definitions, we can shed some light on the intuition behind the statement of Conjecture 1.23. Clearly, on $V^\vee$ we have a distinguished set of B-branes which are the points (with trivial bundle and connection). This set does not depend on the complex structure of $V^\vee$ at all. This suggests that there should be a distinguished set of A-branes on V with little dependence on the symplectic structure. Special Lagrangian branes are a good candidate since they are unaffected by scaling the symplectic form. Thus, there has to be a family of special Lagrangian branes on V parametrised by (the points of) $V^\vee$. Moreover, since points in $V^\vee$ can be deformed to one another (by connecting them through a path), the resulting family of A-branes should all be deformations of one of them. This, together with a theorem of McLean, implies that they form a torus fibration (which can only be nonsingular generically, otherwise the topology of V would be constrained).

Let us conclude the discussion about general aspects of mirror symmetry by mentioning one last form, **homological mirror symmetry**, due to Kontsevich [87]. It can be related to the point of view of branes as follows. First, it is convenient to simplify the problem by relaxing the metric aspects of the theory. This is done by performing a *topological twist*, which is an operation that alters the theory in such a way that one of the supercharge operators Q becomes metric-independent and squares to zero. This allows to pass to the cohomology of this operator, yielding a *topological string theory* which no longer depends on the target space metric.

Depending on one of two choices of topological twist, the result is either called the A**-model** or the B**-model**. Branes which are preserved under this passage to cohomology (i.e. those whose structure is compatible with Q) descend to the topological theory. For the A-model these are precisely the A-branes, although the metric condition (the vanishing of $\mathrm{Im}(\Omega)$) is no longer relevant. For the B-model they are precisely the B-branes, once again without the metric condition (the constant central curvature). These should still be interchanged by mirror symmetry.

The resulting sets of A-branes and B-branes should naturally have the structure of a category: strings constrained between two given branes (more precisely, the space of states of these strings) represent morphisms between the branes, and interactions between a string from a brane L to L' with a string from L' to L'' provide the composition rule. Therefore, we should expect an equivalence of categories between a *category of A-branes* and a *category of B-branes*.

For B-branes, which correspond to connections on vector bundles supported in subvarieties, a natural candidate where they belong to is the (bounded) derived category of coherent sheaves $D^b(V^\vee)$. For A-branes, at least of Lagrangian type, Fukaya [48] proposed a category $\mathrm{Fuk}(V)$ whose objects are Lagrangian submanifolds and whose morphisms are given by Floer cohomology. Technically, it has the structure of a A_∞-category rather than a honest category. However, we can take its derived category and then pass to its cohomology (roughly speaking, we consider morphisms up to homotopy), which results in an actual triangulated category which we will also denote by $\mathrm{Fuk}(V)$. Then, we can state the equivalence of branes as follows.

Conjecture 1.28 (Homological mirror symmetry) *If (V,ω) and $(V^\vee,\omega^\vee)$ are a mirror pair, there is an equivalence of categories*

$$\mathrm{Fuk}(V) \simeq D^b(V^\vee).$$

Remark 1.29 *The metric conditions that we removed by passing to the topological theory can be regarded as stability conditions on both categories $\mathrm{Fuk}(V)$ and $D^b(V^\vee)$. This motivated the introduction [30] of a general notion of stability for triangulated categories which particularises to the previous ones, called Bridgeland stability, building on previous notions such as Π-stability [41].*

Now that we overviewed the relevant mirror symmetry aspects for Calabi–Yau varieties with a Kähler class (V,ω), let us discuss the implications of considering a stronger structure: a hyperkähler manifold V. This is particularly relevant as, recalling Section 1.2, our space of interest $\mathcal{M}(G)$ is hyperkähler. Mathematically, it is a natural question to ask: such a hyperkähler manifold V has a sphere of complex structures with three independent directions I,J,K, and respective Kähler forms $\omega_I,\omega_J,\omega_K \in \Omega^2(V)$. Therefore, we get natural complex symplectic forms

$$\Omega_I := \omega_J + i\omega_K$$

$$\Omega_J := \omega_K + i\omega_I$$

$$\Omega_K := \omega_I + i\omega_J$$

such that each one is holomorphic with respect to the complex structure indicated in the subscript. In particular V is Calabi–Yau in any of the complex structures, so we can consider all the aforementioned mirror symmetry questions by fixing any of them.

However, in this setting it is more interesting to look at how all the structures interact. This analysis was carried out precisely for $\mathcal{M}(G)$ following a physical point of view by Kapustin and Witten [83]. In order to get hyperkähler manifolds to appear naturally in the target space, they start with a *Yang–Mills, four dimensional $N = 4$ supersymmetric gauge theory.* This means that instead of defining fields on a two-dimensional worldsheet Σ, they exist on a four dimensional manifold, and there are $N = 4$ independent generators for the supersymmetry. Moreover, it is a gauge theory and not a sigma model, meaning that fields are not maps to a target space, but rather sections (or connections) of G-bundles on the four-dimensional manifold, where G is the symmetry group of the theory. As with the case of two-dimensional $N = (2, 2)$ SCFTs from before, this theory is not rigorously defined mathematically, but it will allow us to extract mathematical consequences about $\mathcal{M}(G)$.

Then, they perform a dimensional reduction by taking the four dimensional space to be a direct product of a Riemann surface C times the worldsheet Σ. This results in a two-dimensional $N = (4, 4)$ supersymmetric sigma model on Σ, similar to the ones considered above but with more supersymmetry generators. This forces the target V to be hyperkähler. As a matter of fact, letting G be the symmetry group (called *gauge group* in physics) of the starting Yang–Mills theory, the target V in this setting is precisely $\mathcal{M}(G)$ with the hyperkähler structure explained in Section 1.2.

This dimensional reduction picture provides a physical explanation to mirror symmetry for $\mathcal{M}(G)$. Indeed, the starting four-dimensional gauge theory is endowed with *S-duality*, an equivalence between the physical theories for G and $G^\vee$. In the abelian case of $G = U(1)$, this duality is called *electromagnetic duality* and it is actually shown [129] rigorously (since the theories can be defined in the abelian case). The work of Kapustin and Witten explains mirror symmetry on $\mathcal{M}(G)$ as a consequence of carrying S-duality through the above stated dimensional reduction.

In this hyperkähler setting, we can require that a brane be adapted to all three structures simultaneously. For example, if a brane (L, E, ∇) satisfies that $L \subseteq V$, E and ∇ are all holomorphic with respect to I,J and K, therefore it is of type B in all three directions, we will say that it is a **BBB-brane**. Similarly, if $L \subseteq V$ is holomorphic with respect to I and Lagrangian with respect to ω_J and ω_K, while ∇ is a flat connection on a smooth vector bundle E over L (therefore E and ∇ are I-holomorphic if we endow E with the $(0, 1)$-part of ∇ as complex structure, due to flatness), we will say that it is a **BAA-brane**. Notice that in this case, with respect to the complex structure J or K, BAA-branes are special Lagrangians. Up to permutation of the directions these are the only possible combinations, since the quaternionic relations between I, J and K do not allow for an odd number of As.

As before, when we perform a topological twist to descend to a topological theory, there is a choice to be made which results in either an A-model or a B-model. However, now we must also choose a direction in the sphere of complex structures corresponding to the Calabi–Yau manifold with which we perform the twist. The above notation for the branes is as expected: a BBB-brane descends to the B-model at I, J and K, while a BAA-brane descends to the B-model at I and to the A-model at J and K. Interestingly, a consequence of the physics of S-duality is that the A-model for $\mathcal{M}(G)$ and B-model for $\mathcal{M}(G^\vee)$ are not swapped for the same complex structure: one must change the complex structures. In other words, the following can be deduced by physical arguments from S-duality.

Conjecture 1.30 *We expect the following equivalences of physical theories:*

- *The A-model for $\mathcal{M}(G)$ in complex structure J is equivalent to the B-model for $\mathcal{M}(G^\vee)$ in complex structure K.*
- *The B-model for $\mathcal{M}(G)$ in complex structure J is equivalent to the A-model for $\mathcal{M}(G^\vee)$ in complex structure K.*

- *The A-model for $\mathcal{M}(G)$ in complex structure I is equivalent to the A-model for $\mathcal{M}(G^\vee)$ in complex structure I.*
- *The B-model for $\mathcal{M}(G)$ in complex structure I is equivalent to the B-model for $\mathcal{M}(G^\vee)$ in complex structure I.*

In particular, the space of BAA-branes in $\mathcal{M}(G)$ should be in correspondence with the space of BBB-branes in $\mathcal{M}(G^\vee)$, the space of ABA-branes in $\mathcal{M}(G)$ in correspondence with that of ABA-branes in $\mathcal{M}(G^\vee)$ and the space of AAB-branes in $\mathcal{M}(G)$ is in correspondence with that of AAB-branes in $\mathcal{M}(G^\vee)$.

Note that, if we forget about the hyperkähler structure, this means that the mirror partner to the (non-compact) Calabi–Yau manifold $(\mathcal{M}(G), \omega_J)$ with complex structure J is $(\mathcal{M}(G^\vee), \omega_K)$ with complex structure K, while there is no natural candidate for a mirror Calabi–Yau manifold for $(\mathcal{M}(G), \omega_I)$ with complex structure I (in the classical sense of Calabi–Yau mirror symmetry). This was the original proposal of Hausel and Thaddeus [73] as we will recall shortly.

Due to the above predictions on the spaces of branes, there have been multiple efforts to identify examples of the different types in $\mathcal{M}(G)$. The following include some of the best-known ones.

- For **BBB-branes**, the simplest example is clearly the trivial line bundle over $\mathcal{M}(G)$. More generally, any hyperholomorphic connection over the whole of $\mathcal{M}(G)$ is a BBB-brane. For example, if $G = G_{\mathrm{ad}}$ is of adjoint type, there is a universal bundle $\mathbb{E} \to \mathcal{M}(G) \times C$ and, given a point $c \in C$ and a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, one can consider a hyperholomorphic bundle $\rho(\mathbb{E}|_{\mathcal{M}(G)\times\{c\}})$. For branes supported in proper, hyperholomorphic subvarieties, the simplest example comes from taking a complex algebraic subgroup $H \subseteq G$ and considering the natural map $\mathcal{M}(H) \to \mathcal{M}(G)$: the image is a hyperholomorphic subvariety. This fits into the more general setting [56, Theorem 8.10] of fixed points in $\mathcal{M}(G)$ under involutions of the form
$$(E, \varphi) \mapsto (\theta(E) \otimes \eta, \theta(\varphi))$$
where $\theta \in \mathrm{Out}_2(G)$ is an outer class of automorphisms of order 2 in G and $\eta \in H^1(C, Z(G))$ is a $Z(G)$-torsor over C. There is also a wide class of actions by larger finite groups on $\mathcal{M}(G)$ whose fixed point locus results in the support of a BBB-brane [7]. Other examples include BBB-branes supported in the singular fibres of h_G given by parabolic subgroups of G [44] or the Dirac–Higgs bundle [22].
- For **BAA-branes**, the simplest examples are the image of the sections of h_G constructed by Hitchin [77] (these are affine spaces so they only support trivial local systems) as well as the fibres $h_G^{-1}(a)$ of the Hitchin map for $a \in \mathcal{A}(G)$. Another important example, which are the main objects of study of this thesis, are the Białynicki-Birula upward flows $W_{\mathcal{E}}^+$ under the natural $\mathbb{C}^\times$-action on $\mathcal{M}(G)$, which we will explain in detail in the next section. There are also BAA-brane supports given by considering real forms $G_{\mathbb{R}} \subseteq G$ and looking at the image of the natural map $\mathcal{M}(G_{\mathbb{R}}) \to \mathcal{M}(G)$, where $\mathcal{M}(G_{\mathbb{R}})$ is the moduli space of $G_{\mathbb{R}}$-Higgs bundles [52] corresponding via non-abelian Hodge theory to representations of $\pi_1(C)$ into $G_{\mathbb{R}}$. As before, this fits into the more general setting [56, Theorem 8.10] of fixed points in $\mathcal{M}(G)$ under involutions of the form
$$(E, \varphi) \mapsto (\theta(E) \otimes \eta, -\theta(\varphi))$$
(note the minus sign!) where once again $\theta \in \mathrm{Out}_2(G)$ is an outer class of automorphisms of order 2 in G and $\eta \in H^1(C, Z(G))$ is a $Z(G)$-torsor over C.
- For **ABA-branes**, Baraglia and Schaposnik [6] construct examples by taking an antiholomorphic involution $f : C \to C$ and considering is fixed point locus under pullback by f on the character variety $\mathcal{R}(G)$. More generally, this action can be postcomposed with any holomorphic involution $\theta \in \mathrm{Aut}_2(G)$, not necessarily the identity.

- For **AAB-branes**, a similar construction as before works, but now applying an antiholomorphic involution of G (besides the antiholomorphic involution of C). Both this type and the previous are described generally by García-Prada and Biswas [20], relating the fixed point loci of these involutions to moduli spaces of *real Higgs bundles* (i.e. where the base curve is real).

One of the key insights of the work of Kapustin and Witten [83] is that the previous mirror symmetry statements, in particular the homological mirror symmetry

$$\mathrm{Fuk}(\mathcal{M}(G),\omega_K) \simeq D^b(\mathcal{M}(G^\vee),J)$$

has as a natural consequence the **geometric Langlands correspondence** [10], a profound relation between algebraic geometry and representation theory which adapts the classical Langlands program from number theory to geometry and it is claimed to have been proven recently in a series of preprints starting with [3]. The first part of the correspondence is an equivalence

$$D^b(\text{D-mod}(\mathrm{Bun}_G)) \simeq D^b(\mathrm{Loc}_{G^\vee})$$

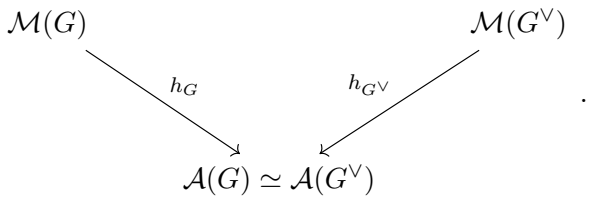
between the derived category of *D-modules* on the stack Bun_G of principal G-bundles over C and the derived category (of coherent sheaves) of $\mathrm{Loc}_{G^\vee}$ the stack of $G^\vee$-local systems on C. Recalling the definition of the complex structure J on $\mathcal{M}(G^\vee)$ (coming from non-abelian Hodge theory) it is easy to identify $(\mathcal{M}(G^\vee),J)$ with the space of $G^\vee$-local systems on C. Kapustin and Witten provide a physical interpretation on how to identify $\mathrm{Fuk}(\mathcal{M}(G),\omega_K)$ with $D^b(\text{D-mod}(\mathrm{Bun}_G))$ by introducing a *canonical coisotropic brane* (which is an ABA-brane, in particular an A-brane in direction K) and arguing that the set of homomorphisms between this canonical brane and a given A-brane (in direction K) should yield a D-module on Bun_G.

The above equivalence of derived categories is only part of the geometric Langlands correspondence. What remains for the statement to be complete (e.g. [39, Conjecture 2.1]) is that, given $c \in C$ and $\mu \in X_*(T)$, natural operations defined on each side of the equivalence from c and μ have to be intertwined. On one hand, we have the **tensorisation functor** $\mathcal{W}_c^\mu : D^b(\mathrm{Loc}_{G^\vee}) \to D^b(\mathrm{Loc}_{G^\vee})$ defined by $\mathcal{F} \mapsto \mathcal{F} \otimes \rho^\mu(\mathbb{F}|_{\mathrm{Loc}_{G^\vee}\times\{c\}})$ where $\rho^\mu : G^\vee \to \mathrm{GL}(V^\mu)$ is the irreducible representation of $G^\vee$ with highest weight μ and $\mathbb{F}$ is the universal local system on $\mathrm{Loc}_{G^\vee} \times C$.

On the other hand, the **Hecke functor** $\mathcal{H}_c^\mu : D^b(\text{D-mod}(\mathrm{Bun}_G)) \to D^b(\text{D-mod}(\mathrm{Bun}_G))$ is given by an integral transform induced by certain kernel (see [39, Conjecture 2.1] for details) on $\mathrm{Hecke}_G^{c,\mu}$, which is the stack parametrising Hecke transformations at c of type μ, that is, triples (E,E',Ψ) where $E,E' \to C$ are principal G-bundles and $\Psi : E|_{C\smallsetminus c} \xrightarrow{\sim} E'|_{C\smallsetminus c}$ is an isomorphism such that, given any irreducible representation $\rho^\nu : G \to \mathrm{GL}(V_\nu)$ of highest weight $\nu \in X^*(T)$, the induced twisted morphism $\rho(\Psi) : E(V_\nu) \to E'(V_\nu) \otimes \mathcal{O}_C(\nu(\mu)\cdot c)$ is an injection. The integral transform is defined with respect to the natural projections to Bun_G given by extracting V or V'. These Hecke transformations are one the main ingredients in Chapter 2 to prove Theorem 2.24, but we do not need to consider the kernel nor the integral transform.

The intertwining of both functors is the natural geometric analogue of the interchange between Hecke eigenvalues and Frobenius eigenvalues from the classical Langlands program relating arithmetic and representation theory (see e.g. [47, Section 2.4]). From the point of view of mirror symmetry, Kapustin and Witten [83] explain this intertwining by interpreting the functors as physical operators on the mirror gauge theories, called **Wilson** and **'tHooft** operators, respectively, which act on each of the relevant spaces of branes and get interchanged by the duality.

So far we have discussed several consequences of mirror symmetry for $\mathcal{M}(G)$ following the circle of ideas of Kapustin and Witten, but it is worth mentioning how they can be geometrically realised: there is a clear picture of SYZ mirror symmetry between $(\mathcal{M}(G),\omega_J)$ and $(\mathcal{M}(G^\vee),\omega_K)$ which was the original proposal of Hausel and Thaddeus [73]. Indeed, the dual fibrations are provided by the Hitchin maps



Note that the bases, which are locally modelled on $\mathfrak{t}/W$ and $\mathfrak{t}^\vee/W$, can be canonically identified using the Killing form to pair $\mathfrak{t}$ with $\mathfrak{t}^\vee$ in a W-equivariant way. Moreover, the generic fibres are abelian varieties by the Beauville–Narasimhan–Ramanan (BNR) correspondence using spectral curves (in the case of classical groups) [8, 76] or via the cameral curve description of Donagi and Gaitsgory [38]. Since h_G is an algebraically integrable system with respect to $\omega = \Omega_I$, the fibres are also holomorphic Lagrangian in $\mathcal{M}(G)$, hence we have dual fibrations by special Lagrangian tori in complex structures J and K, as required by the SYZ proposal.

The duality of the fibres (which in this case can be stated precisely as a duality of abelian varieties) was shown for $G = \mathrm{SL}_n(\mathbb{C})$ by Hausel and Thaddeus [73] and for arbitrary semisimple G by Donagi and Pantev [39]. We can summarise it as follows:

Theorem 1.31 (SYZ mirror symmetry for $\mathcal{M}(G)$ [39, Theorems A and B]). *For $a \in \mathcal{A}(G) \smallsetminus \Delta_G \simeq \mathcal{A}(G^\vee) \smallsetminus \Delta_{G^\vee}$ where Δ_G and $\Delta_{G^\vee}$ are the ramification divisors, we have:*

- *The connected components of the fibre $h_G^{-1}(a)$ are indexed by $\pi_1(G)$ and they correspond to the intersection of the fibre with each of the connected components of $\mathcal{M}(G)$.*
- *The identity components of the fibres $h_G^{-1}(a)$ and $h_{G^\vee}^{-1}(a)$ are dual abelian varieties. The remaining components are torsors under the identity component, so choosing an origin identifies them as dual abelian varieties.*
- *The identity components of $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$ are dual families of abelian varieties over $\mathcal{A}(G) \smallsetminus \Delta_G$.*
- *Considering all connected components, the moduli spaces $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$ over $\mathcal{A}(G) \smallsetminus \Delta_G$ are dual gerbes.*
- *The previous duality of gerbes intertwines the natural operators on each side induced (see [39, Appendix]) from the tensorisation and Hecke operators explained above.*

The appearance of gerbes when accounting for all connected components is due to the fact that in general there is no way to choose a family over $\mathcal{A}(G) \smallsetminus \Delta_G$ of origins for the torsors, so one should not simply expect a duality of abelian varieties in families. For example, the sections of h_G constructed by Hitchin [77] lie in the identity component.

This duality also provides an equivalence of derived categories over $\mathcal{A}(G) \smallsetminus \Delta_G$ by applying a Fourier–Mukai transform as introduced by Mukai [99]. In general, these transforms between $D^b(X)$ and $D^b(Y)$ are induced by an element $\mathcal{K} \in D^b(X \times Y)$ by mapping $\mathcal{F} \mapsto (\pi_Y)_*(\pi_X^*\mathcal{F} \otimes \mathcal{K})$. In $\mathcal{M}(G)$, the previous theorem allows us to perform it fibrewise by taking $\mathcal{F}$ to be the Poincaré line bundle on the product of an abelian variety times its dual (i.e. the universal line bundle), and this can be extended in families [39, Corollary 5.5] over $\mathcal{A}(G) \smallsetminus \Delta_G$. In short, a fibrewise Fourier–Mukai transform is expected to induce an equivalence of categories

$$D^b(\mathcal{M}(G)) \simeq D^b(\mathcal{M}(G^\vee)).$$

Donagi and Pantev interpret this as a *classical limit* of the geometric Langlands correspondence explained above, in the sense that, roughly speaking, taking the associated graded of a D-module

over Bun_G results in a G-Higgs bundle. As for mirror symmetry, this can be thought as the equivalence of the spaces of B-branes in complex structure I predicted by Kapustin and Witten, and therefore it is natural to expect that it identifies more generally BAA-branes with BBB-branes.

Finally, $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$ also satisfy (partly conjecturally) topological mirror symmetry. This was proposed by Hausel and Thaddeus [73], and some modifications to the standard statement for compact, smooth Calabi–Yau manifolds must be performed in order to account for the non-compactness and the non-smoothness (they work with orbifold moduli spaces meaning that, while there are no strictly semistable points, there are some stable Higgs bundles with larger automorphism group which cannot be avoided) of the spaces.

More precisely, in their proposal they work with $\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C}))$, where the superscript d indicates that they consider *twisted* $\mathrm{SL}_n(\mathbb{C})$-Higgs bundles, in the sense that the corresponding vector bundle satisfies $\det E \simeq \mathcal{O}_C(cd)$ for a fixed $c \in C$. This allows to obtain a smooth space by choosing $(d,n) = 1$. For the dual, they consider $\mathcal{M}^d(\mathrm{PGL}_n(\mathbb{C}))$ which is the connected component for $d \in \pi_1(\mathrm{PGL}_n(\mathbb{C})) \simeq \mathbb{Z}/n\mathbb{Z}$ and, when $(d,n) = 1$, it is an orbifold.

Due to the non-compactness of the spaces, topological mirror symmetry involves the mixed Hodge structure on the cohomology of the spaces and becomes a statement regarding the *E-polynomial*

$$E(\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C}))) = \sum_{p,q} h^{p,q}(\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C})))u^p v^q$$

where the Hodge number $h^{p,q}(\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C})))$ is computed as an alternating sum of the mixed Hodge numbers of $\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C}))$.

For the orbifold $\mathcal{M}^d(\mathrm{PGL}_n(\mathbb{C})) = \mathcal{M}^d(\mathrm{SL}_n(\mathbb{C}))/\Gamma$ even more adjustments need to be made: the E-polynomial has to be defined equivariantly with respect to the finite group action Γ and then some *stringy* adjustments need to be made, namely accounting for a *fermionic shift* as well as for a *B-field* which is defined as a $U(1)$-gerbe $B^\vee$ over $\mathcal{M}^d(\mathrm{PGL}_n(\mathbb{C}))$ recording the liftings of the universal $\mathrm{PGL}_n(\mathbb{C})$-bundle to a $\mathrm{SL}_n(\mathbb{C})$-bundle. Without going into details, the outcome is a *stringy E-polynomial* $E_{st}^{B^\vee}(\mathcal{M}^d(\mathrm{PGL}_n(\mathbb{C})))$. Then, the topological mirror symmetry proposal is

Theorem 1.32 (Topological mirror symmetry for $\mathcal{M}(\mathrm{SL}_n(\mathbb{C}))$ [64,73]). *For $(d,n) = 1$, there is an agreement*

$$E(\mathcal{M}^d(\mathrm{SL}_n(\mathbb{C}))) = E_{st}^{B^\vee}(\mathcal{M}^d(\mathrm{PGL}_n(\mathbb{C}))).$$

In their original proposal, Hausel and Thaddeus proved it [73, Theorem 10.6] for $n = 2, 3$ and later Gröchenig, Wyss and Ziegler proved it for all n via arithmetic techniques [64, Theorem 1.1]. For other semisimple groups G there is not yet such a proven statement.

1.4 Fixed points of the $\mathbb{C}^\times$-action in $\mathcal{M}(G)$

The moduli space $\mathcal{M}(G)$ is equipped with a natural action of $\mathbb{C}^\times$, where $\lambda \in \mathbb{C}^\times$ acts by scaling the Higgs field:

$$(E, \varphi) \mapsto (E, \lambda\varphi).$$

Notice that the Hitchin map h_G from Definition 1.17 is $\mathbb{C}^\times$-equivariant with respect to the $\mathbb{C}^\times$-action on $\mathcal{A}(G)$ with weight $\deg p_i$ on the $H^0(C, K_C^{\deg p_i})$ factor. Similarly, the holomorphic symplectic form $\omega \in H^0(\Omega^2_{\mathcal{M}^s(G)})$ is scaled with weight one under this action:

$$\omega \mapsto \lambda \cdot \omega,$$

as can be seen from its definition in terms of the deformation complex in Section 1.2.

In the present section we analyse the locus of fixed points for this $\mathbb{C}^\times$-action [14, Section 4.2] (see also [113]). These are closely related to $\mathbb{Z}$-gradings of Lie algebras.

Definition 1.33 *A* $\mathbb{Z}$***-grading*** *of the Lie algebra* $\mathfrak{g}$ *is a direct sum decomposition as vector spaces*

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

Given such a grading, the piece $\mathfrak{g}_0 \subseteq \mathfrak{g}$ is a Lie subalgebra and hence it has a corresponding connected subgroup $G_0 \subseteq G$. Since $\mathfrak{g}$ is semisimple, there exists a **grading element** $\zeta \in \mathfrak{g}_0$ such that

$$\mathfrak{g}_j = \{X \in \mathfrak{g} : [\zeta, X] = jX\}.$$

We have the description $G_0 = C_G(\zeta)$ as the centraliser of ζ in G. In order to see this, it suffices to justify that the centraliser of a semisimple element of $\mathfrak{g}$ under the adjoint action of G is connected. Let $S \subseteq G$ be the torus with Lie algebra $\langle\zeta\rangle \subseteq \mathfrak{g}$, which exists because ζ is semisimple with integral eigenvalues. Then, we have $C_G(\zeta) = C_G(S)$. Now, the centraliser of a torus is always connected if G is [78, Theorem 22.3], and we conclude. Furthermore, the fact that $[\mathfrak{g}_0, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$ implies that G_0 acts on each $\mathfrak{g}_j$ by the adjoint representation. These representations $(G_0, \mathfrak{g}_j)$, whenever $j \neq 0$, are called **Vinberg** $\mathbb{C}^\times$**-pairs** [49, 50], since they were studied by Vinberg [126].

The $\mathbb{Z}$-gradings of $\mathfrak{g}$ can be classified up to conjugacy by noting that ζ can be conjugated into $\mathfrak{t}$ and must have integral eigenvalues. After conjugation by a Weyl group element, we may assume that the eigenvalues on simple root spaces are not negative. Therefore, it is determined by $\beta_j(\zeta) \in \mathbb{Z}_{\geqslant 0}$ for each $\beta_j \in \Pi$, so that the classification is given by labellings of the vertices of the Dynkin diagram of $\mathfrak{g}$ with non-negative integers. Moreover, in order to classify the different possible parabolic subalgebras $\mathfrak{p} = \bigoplus_{j\geqslant 0} \mathfrak{g}_j$ up to conjugation, it suffices to consider labellings with zeroes or ones, as explained in Section 1.1.

Definition 1.34 *Let* $\hat{G} \subseteq G$ *be a subgroup and* $V \subseteq \mathfrak{g}$ *be a subspace which is invariant under the adjoint representation of* G *restricted to* $\hat{G}$. *A* G*-Higgs bundle* (E, φ) ***reduces to a*** $(\hat{G}, V)$***-Higgs pair*** *if there exists a reduction of structure group* $\sigma \in H^0(C, E(G/\hat{G}))$ *of* E *to* $\hat{G}$ *such that* $\varphi \in H^0(C, E_\sigma(V) \otimes K_C) \subseteq H^0(C, E(\mathfrak{g}) \otimes K_C)$.

As explained in Remark 1.13, there exists a notion of polystability for $(G_0, \mathfrak{g}_j)$-Higgs pairs, and we have the following result (see e.g. [14, Proposition 4.15]).

Proposition 1.35. *A polystable* G*-Higgs bundle* $(E, \varphi) \in \mathcal{M}(G)$ *is fixed by the* $\mathbb{C}^\times$*-action if and only if there exists a* $\mathbb{Z}$*-grading* $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ *such that* (E, φ) *reduces to a polystable* $(G_0, \mathfrak{g}_j)$*-Higgs pair for* $j \neq 0$.

In other words, the fixed point locus $\mathcal{M}(G)^{\mathbb{C}^\times}$ can be described as the union of the images for the natural maps

$$\mathcal{M}(G_0, \mathfrak{g}_j) \to \mathcal{M}(G)$$

obtained by considering all the different Vinberg $\mathbb{C}^\times$-pairs $(G_0, \mathfrak{g}_j)$ of G. We say that fixed points in the image of $\mathcal{M}(G_0, \mathfrak{g}_j)$ are of **type** $(G_0, \mathfrak{g}_j)$.

Notice that, as can also be deduced from the $\mathbb{C}^\times$-equivariance of h_G, the above implies that fixed points are nilpotent (as there are finitely many pieces in the grading). Of course, there are nilpotent Higgs bundles which are not $\mathbb{C}^\times$-fixed. For example, those that reduce to $(G_0, \bigoplus_{j>0} \mathfrak{g}_j)$-Higgs pairs but do not reduce further to any $(G_0, \mathfrak{g}_j)$-Higgs pair.

Example 1.36 *Consider the case of $G = \mathrm{SL}_n(\mathbb{C})$. Via the standard representation, a $\mathrm{SL}_n(\mathbb{C})$-Higgs bundle can be seen as a pair (E, φ) where E is a vector bundle of rank n and trivial determinant over C and $\varphi : E \to E \otimes K_C$ is a traceless twisted bundle endomorphism. In this case, it is known (as shown by Hitchin [75, Section 7] and Simpson [115, Section 4], see also [73, Lemma 9.2] and [70, Section 3.1]) that fixed points are those that admit a direct sum splitting*

$$E = E_0 \oplus \cdots \oplus E_{m-1},$$

of vector subbundles $E_k \subseteq E$ with $\operatorname{rank} E_k = r_k$, $\sum_{k=0}^{m-1} r_k = n$ and such that $\varphi(E_k) \subseteq E_{k+1} \otimes K_C$. These objects and their moduli spaces were studied in depth in [53, 54]. In our setting, this corresponds to the $\mathbb{Z}$-grading of $\mathfrak{sl}_n(\mathbb{C})$ given by choosing a splitting $\mathbb{C}^n = V_0 \oplus \cdots \oplus V_{m-1}$ with $\dim V_k = r_k$ and letting

$$\mathfrak{g}_p := \{A \in \mathfrak{sl}_n(\mathbb{C}) : A(V_k) \subseteq V_{p+k}\}.$$

According to the classification of $\mathbb{Z}$-gradings of $\mathfrak{sl}_n(\mathbb{C})$ these are all the possible ones, which is consistent with the classical fact that there are no more fixed points than those explained above.

Since, in this case, the grading is determined by the ordered partition $(r_0, \ldots, r_{m-1})$ of n, it is customary to name these fixed points of ***type*** *$(r_0, \ldots, r_{m-1})$.*

In what follows we assume, without loss of generality, that the Vinberg pairs under consideration are of the form $(G_0, \mathfrak{g}_1)$. Indeed, if $j \neq 1$, we may consider instead the lie subalgebra $\bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_{pk}$ and rescale its grading. Furthermore, as we will see shortly, the fixed point types of interest for our work are all of the form $(G_0, \mathfrak{g}_1)$ without any rescaling involved.

We start by remarking that the trivial grading $\mathfrak{g} = \mathfrak{g}_0$ may be considered. In this case, $\mathfrak{g}_1 = 0$ and $\mathcal{M}(G_0, \mathfrak{g}_1) = \mathcal{N}(G)$ is the moduli space of polystable principal G-bundles, where a principal G-bundle E is mapped into $\mathcal{M}(G)$ as $(E, 0)$. These are certainly fixed by the $\mathbb{C}^\times$-action.

Now we describe fixed points with *generically regular* Higgs field, that is, such that the centraliser in G of the Higgs field has (generically) the minimum possible dimension, equal to $r = \operatorname{rank} \mathfrak{g}$. In the case of $G = \mathrm{SL}_n(\mathbb{C})$ from the example above, this corresponds to the situation where every $r_j = 1$, known in the literature as *fixed points of type* $(1, 1, \ldots, 1)$. These are the fixed points that were considered in [70] in the context of very stable Higgs bundles. We now define the analogue for G-Higgs bundles. In terms of Vinberg pairs, there is only one that contains regular elements as an open subset of $\mathfrak{g}_1$.

Definition 1.37 *The* ***Borel grading*** *of $\mathfrak{g}$ is given by $\mathfrak{g}_0 := \mathfrak{t}$ and*

$$\mathfrak{g}_j := \bigoplus_{\beta\in\Delta:\mathrm{ht}(\beta)=j} \mathfrak{g}_\beta$$

for $j \neq 0$, where $\mathrm{ht}(\beta)$ denotes the height of β with respect to Π. Equivalently, it is given by the grading element $\zeta = \sum_{j=1}^{r} \omega_j^\vee$.

The terminology refers to the fact that every $\mathbb{Z}$-grading corresponds to a parabolic subalgebra $\mathfrak{p} := \bigoplus_{j\geqslant 0} \mathfrak{g}_j$, and in the Borel grading the corresponding subalgebra is precisely the Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$.

Remark 1.38 *In the Borel grading, we have that $G_0 = T$ and that the degree one part is the sum of the simple root spaces, i.e. $\mathfrak{g}_1 = \bigoplus_i \mathfrak{g}_{\beta_i}$. The G_0-action on $\mathfrak{g}_1$ has an open orbit $\Omega \subseteq \mathfrak{g}_1$ given by elements $e = \sum_i n_i X_{\beta_i}$ with $n_i \neq 0$ for all i, which are the regular nilpotents of $\mathfrak{g}$ belonging to $\mathfrak{g}_1$. The elements of degree greater than one are not regular. Similarly, any other choice of grading does not have homogeneous regular nilpotent elements of nonzero degree.*

Indeed, suppose that $e \in \mathfrak{g}_k$ is regular nilpotent with $k \neq 0$. Extend it to an $\mathfrak{sl}_2(\mathbb{C})$-triple (e, h, f) by the Jacobson–Morozov theorem. Then we have that $h - \frac{2}{k}\zeta$ centralises e, so it must be nilpotent (see e.g. [89, Proposition 14 and Remark 15]). However, both h and ζ are semisimple, so $h - \frac{2}{k}\zeta = 0$ and $\zeta = \frac{kh}{2}$. Since all principal $\mathfrak{sl}_2$-triples are conjugate, this means that we may take $\zeta = k\sum_{j=1}^{r}\omega_j^\vee$ up to conjugation, because $2\sum_{j=1}^{r}\omega_j^\vee$ is the semisimple element of the principal $\mathfrak{sl}_2(\mathbb{C})$-triple where e is equal to the sum of any chosen set of simple root vectors. Thus, the resulting grading is the Borel grading up to scaling.

Thus we will focus on the fixed point components given by the Borel grading and such that the Higgs field has degree one.

Definition 1.39 *A fixed point $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ is said to be of **Borel type** if it reduces to a $(G_0, \mathfrak{g}_1)$-Higgs pair for the Borel grading of $\mathfrak{g}$ and φ belongs to the open G_0-orbit $\Omega \subseteq \mathfrak{g}_1$ generically.*

As an illustration, we can write fixed points of Borel type in terms of vector bundles for the classical groups.

Example 1.40 *If $G = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{PGL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$, fixed points of Borel type can be regarded as vector bundles $E = L_1 \oplus \cdots \oplus L_n$ where $\operatorname{rank} L_j = 1$ with Higgs field given by $\varphi = (\varphi_j)_j$ where $\varphi_j \in H^0(L_j^* L_{j+1} K_C)$, i.e. a choice of section for each simple root (notice here that we choose T to be the diagonal matrices with coordinates e_j, and $e_{j+1} - e_j$ to be the positive roots, which is the opposite of some conventions in the literature). These sections must not be identically zero in order for the Higgs field to be generically in the open orbit.*

Example 1.41 *If $G = \mathrm{SO}_{2n}(\mathbb{C})$, where for convenience we think of the bilinear quadratic form as $Q = (\delta_{i,2n-j+1})_{i,j}$ (here δ_{ab} is the Dirac delta), fixed points of Borel type are given by vector bundles*

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

with $\operatorname{rank} L_j = 1$ and the Higgs field is determined by nonzero sections $\varphi_j \in H^0(L_j^* L_{j+1} K_C)$ for $j \in \{1, \ldots, n-1\}$ (every one of these corresponds to the simple roots $-e_j + e_{j+1}$) and $\varphi_n \in H^0(L_{n-1}^* L_n^* K_C)$ (corresponding to the simple root $-e_{n-1} - e_n$). The Higgs field can be then written as

$$\varphi = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ \varphi_1 & \ddots & & & & \\ & \ddots & 0 & & & \\ & & \varphi_{n-1} \; 0 & & & \\ \hline & & \varphi_n \; 0 & 0 & & \\ & & & -\varphi_n^t & -\varphi_{n-1}^t & \ddots \\ & & & & \ddots & 0 \\ & & & & & -\varphi_1^t \; 0 \end{array}\right).$$

Example 1.42 *Similarly, for $G = \mathrm{SO}_{2n+1}(\mathbb{C})$ with bilinear quadratic form $Q = (\delta_{i,2n-j+2})_{i,j}$, fixed points of Borel type are*

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus \mathcal{O}_C \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

$\operatorname{rank} L_j = 1$ and the Higgs field is determined by nonzero sections $\varphi_j \in H^0(L_j^* L_{j+1} K_C)$ for $j \in \{1, \ldots, n-1\}$ for the simple roots $-e_j + e_{j+1}$ and $\varphi_n \in H^0(L_n^* K_C)$ (corresponding to $-e_n$). The Higgs field can be written as

$$\varphi = \begin{pmatrix} 0 & & & & & & & & & \\ \varphi_1 & \ddots & & & & & & & & \\ & \ddots & 0 & & & & & & & \\ & & \varphi_{n-1} & 0 & & & & & & \\ & & & \varphi_n & 0 & & & & & \\ & & & & -\varphi_n^t & 0 & & & & \\ & & & & & -\varphi_{n-1}^t & \ddots & & & \\ & & & & & & \ddots & 0 & & \\ & & & & & & & -\varphi_1^t & 0 \end{pmatrix}.$$

Example 1.43 *Finally we consider $G = \mathrm{Sp}_{2n}(\mathbb{C})$ with symplectic form $Q = (\varepsilon_{i,j}\delta_{i,2n-j+1})_{i,j}$ where*

$$\varepsilon_{i,j} = \begin{cases} 1 & i \leqslant j \\ -1 & i > j \end{cases}.$$

Fixed points of Borel type are

$$E = (L_1 \oplus \cdots \oplus L_n) \oplus (L_n^* \oplus \cdots \oplus L_1^*),$$

and the Higgs field is determined by nonzero sections $\varphi_i \in H^0(L_j^* L_{j+1} K_C)$ for $j \in \{1, \ldots, n-1\}$ corresponding to the simple roots $-e_j + e_{j+1}$ and $\varphi_n \in H^0((L_n^2)^* K_C)$ corresponding to $-2e_n$. The Higgs field can be written as

$$\varphi = \left(\begin{array}{ccccc|ccccc} 0 & & & & & & & & & \\ \varphi_1 & \ddots & & & & & & & & \\ & \ddots & 0 & & & & & & & \\ & & \varphi_{n-1} & 0 & & & & & & \\ \hline & & & \varphi_n & 0 & & & & & \\ & & & & -\varphi_{n-1}^t & \ddots & & & & \\ & & & & & \ddots & 0 & & & \\ & & & & & & -\varphi_1^t & 0 \end{array}\right).$$

The components φ_j of the Higgs field that appeared in the examples can be recovered generally as follows.

Definition 1.44 *Let (E, φ) be a fixed point of Borel type. For each simple root $\beta_j \in \Pi$, we define the j-**th component** of (E, φ) as:*

$$\varphi_j := \pi_j(\varphi),$$

where

$$\pi_j : H^0(E_T(\mathfrak{g}_1) \otimes K_C) \to H^0(E_T(\mathfrak{g}_{\beta_j}) \otimes K_C)$$

is the natural projection.

Since $\mathfrak{g}_1$ is the sum of simple root spaces, the components recover the Higgs field. Moreover, notice that the $E(\mathfrak{g}_\beta)$ are line bundles since $\dim \mathfrak{g}_\beta = 1$. Notice that the requirement that φ generically belongs to the open G_0-orbit in $\mathfrak{g}_1$ translates to all the components being nonzero.

We can naturally record the information of our fixed point type using a divisor valued in the set $P^+_{\Delta^\vee}$ of dominant coweights.

Definition 1.45 *Let (E,φ) be a fixed point of Borel type and $c \in C$. The **multiplicity coweight** of (E,φ) at c is defined to be*

$$\mu_c(E,\varphi) := \sum_{j=1}^{r} \mathrm{ord}_c(\varphi_j)\omega_j^\vee,$$

where ord_c denotes the order of vanishing at c of the component φ_j.

*The **multiplicity divisor** of (E,φ) is defined as*

$$\mu(E,\varphi) := \sum_{c\in C} \mu_c(E,\varphi)\cdot c.$$

Note that the multiplicity divisor is indeed valued in $P^+_{\Delta^\vee}$ since $\mathrm{ord}_c(\varphi_j) \geqslant 0$ due to the sections being holomorphic.

Remark 1.46 *Recall that we can associate to the G_0-bundle E a **topological type** $d_E \in \pi_1(G_0) \simeq X_*(T)$, uniquely determined by the condition*

$$\deg \chi(E) = \chi(d_E)$$

for every character $\chi \in X^*(T)$. This topological type is encoded in the degree $\deg(\mu(E,\varphi)) := \sum_{c\in C}\mu_c(E,\varphi) \in P_{\Delta^\vee}$. Indeed, the definition implies that $E(\mathfrak{g}_{\beta_j}) \otimes K_C \simeq \mathcal{O}_C(\sum_{c\in C}\beta_j(\mu_c(E,\varphi)))$, which automatically forces

$$d_E = \deg(\mu(E,\varphi)) + \mu_{\mathrm{can}}$$

where $\mu_{\mathrm{can}} = -(2g-2)\zeta = -(2g-2)\sum_{j=1}^r \omega_j^\vee$. In particular, since $\mu_{\mathrm{can}} \in Q_{\Delta^\vee}$ and $d_E \in X_*(T)$, this implies that $\deg(\mu(E,\varphi)) \in X_*(T)$.

Remark 1.47 *More strongly, the information of $\mu(E,\varphi)$ allows to recover $(E/Z(G),\varphi)$, i.e the starting Higgs bundle (E,φ) can be determined up to a twist by a $Z(G)$-bundle (therefore uniquely if G is of adjoint type). Indeed, the coweights $\mu_c(E,\varphi)$ fully determine the line bundles $E(\mathfrak{g}_{\beta_j})$ and their sections $\pi_j(\varphi)$. Since the simple roots span the root lattice, knowing $E(\mathfrak{g}_{\beta_j})$ recovers $E/Z(G)$ and therefore E is recovered up to a twist of a $Z(G)$-bundle. The Higgs field is completely recovered from the $\pi_j(\varphi)$. On the other hand, it is not always possible to construct (E,φ) with arbitrarily prescribed $\mu(E,\varphi)$ valued in $P^+_{\Delta^\vee}$. For example, Remark 1.46 shows that one must have $\deg(\mu(E,\varphi)) \in X_*(T)$.*

Definition 1.48 *Let (E,φ) be a smooth $\mathbb{C}^\times$-fixed point of Borel type. We say that its multiplicity divisor $\mu(E,\varphi)$ is **reduced** if and only if $\mu_c(E,\varphi)$ is minuscule for all $c \in C$.*

We conclude by analysing the smoothness in $\mathcal{M}(G)$ of a fixed point of Borel type. First, note that for this kind of Higgs bundles, stability is equivalent to smoothness as a point in the moduli space.

Proposition 1.49. *Let (E,φ) be a stable $\mathbb{C}^\times$-fixed point of Borel type. Then, (E,φ) is simple. In particular, $(E,\varphi) \in \mathcal{M}(G)^s$.*

Proof. This follows from the fact that, when (E,φ) is a fixed point of Borel type, the Higgs field φ takes values in a regular nilpotent orbit generically over C. The centraliser of a regular nilpotent element is of the form $Z(G)\cdot U$ where $U \subseteq G$ is a connected abelian unipotent subgroup with

$\dim(U) = \operatorname{rank}(\mathfrak{g})$ (see e.g. [89, Proposition 14 and Remark 15]). Therefore, its only finite order elements are those of $Z(G)$.

Now, if $f \in \operatorname{Aut}(E,\varphi) \simeq H^0(C, E(G))$ is an automorphism, which we regard as a G-equivariant section $f : E \to G$ (with respect to the left conjugation action of G on G), the compatibility with the Higgs field is $\operatorname{Ad}_f(\varphi) = \varphi$. In other words, generically over C, the automorphism f takes values in the centraliser of a regular nilpotent element.

By [52, Proposition 3.11], the stability of (E,φ) implies that $\operatorname{Aut}(E,\varphi)$ is finite. Therefore, f is a finite order element and must take values generically in $Z(G)$. This implies that f is constant with values in $Z(G)$ over the whole of C, as $Z(G)$ is discrete. □

Therefore, it suffices to assess the stability, which can be cleanly rewritten in terms of the multiplicity divisor.

Proposition 1.50. *Let (E,φ) be a $(G_0,\mathfrak{g}_1)$-Higgs bundle for the Borel grading such that φ belongs to the open G_0-orbit $\Omega \subseteq \mathfrak{g}_1$ generically. Let $d_E = \deg(\mu(E,\varphi)) + \mu_{\mathrm{can}} \in X_*(T)$ denote the topological type from Remark 1.46. The following are equivalent:*

1. *(E,φ) is stable as a G-Higgs bundle.*
2. *For all nontrivial $\chi \in X^*_+(T) \smallsetminus \{0\}$ we have $\chi(d_E) < 0$.*
3. *For all $k \in \{1,\dots,r\}$ we have $\omega_k(d_E) < 0$.*
4. *If $A = (a_{jk})_{j,k}$ denotes the Cartan matrix of $\mathfrak{g}$, with inverse $A^{-1} = (b_{jk})_{j,k}$, then*

$$\frac{\sum_{j=1}^r b_{jk}\beta_j(\deg(\mu(E,\varphi)))}{\sum_{j=1}^r b_{jk}} < 2g-2$$

holds for all $k \in \{1,\dots,r\}$.

The previous equivalences also hold replacing stable by semistable and strict inequalities by non-strict inequalities.

Proof. First we will see that conditions 2 through 4 are equivalent Lie theoretic statements. The equivalence of 2 and 3 follows from the fact that dominant characters of T are spanned with non-negative coefficients by the fundamental weights (and every fundamental weight has a multiple which lifts to a character of T). The equivalence of 3 and 4 follows from computing

$$\omega_k(d_E) = \omega_j(\mu(E,\varphi) + \mu_{\mathrm{can}})$$

using that

$$\omega_k(\omega_j^\vee) = \left(\sum_{m=1}^r b_{mk}\beta_m\right)(\omega_j^\vee) = b_{jk}.$$

Now let $P \subseteq G$ be a standard parabolic subgroup and $\chi \in X^*_+(P) \smallsetminus \{0\}$ a nontrivial dominant character. Let $\sigma \in H^0(C, E(G/P))$ be a reduction of structure group such that $\varphi \in H^0(E_\sigma(\mathfrak{p}) \otimes K_C)$. Since φ is generically a regular nilpotent element of $\mathfrak{g}$, Lemma 1.1 implies that σ is uniquely determined over the dense open subset $U \subseteq C$ such that $\varphi|_U$ is regular. Then, properness of G/P implies that σ is uniquely determined everywhere. Now we will see that indeed such a σ exists: we know that the structure group E reduces to the maximal torus G_0 in such a way that $\varphi \in H^0(E_{G_0}(\mathfrak{g}_1) \otimes K_C)$, therefore extending structure group via $G_0 \subseteq P$ and $\mathfrak{g}_1 \subseteq \mathfrak{b} \subseteq \mathfrak{p}$ gives the desired reduction.

From the construction of σ, we see that its degree can be obtained from the topological type of the reduced principal G_0-bundle, in other words

$$\deg(E, \sigma, \chi) = \chi(d_E).$$

The stability of (E, φ) is equivalent to this quantity being negative for every standard parabolic $P \subseteq G$ and $\chi \in X^*_+(P) \smallsetminus \{0\}$, which in turn is equivalent to condition 2. □

Corollary 1.51. *Let (E, φ) be a stable $\mathbb{C}^\times$-fixed point of Borel type. Suppose that (E', φ') is a $(G_0, \mathfrak{g}_1)$-Higgs pair for the Borel grading such that φ belongs to the open G_0-orbit $\Omega \subseteq \mathfrak{g}_1$ generically. Suppose that $\mu(E', \varphi') = \mu(E, \varphi) - \beta^\vee \cdot c$ for some $c \in C$ and $\beta \in \Delta^+$. Then, (E', φ') is stable.*

Proof. Follows from the second condition of Proposition 1.50 as, given $\chi \in X^*_+(T) \smallsetminus \{0\}$, we have

$$\chi(d_{E'}) = \chi(d_E) - \chi(\beta^\vee) \leqslant \chi(d_E) < 0$$

since $\chi(\beta^\vee) \geqslant 0$. □

Corollary 1.52. *Let (E, φ) be a stable $\mathbb{C}^\times$-fixed point of Borel type. Suppose that (E', φ') is a $(G_0, \mathfrak{g}_1)$-Higgs pair for the Borel grading such that φ belongs to the open G_0-orbit $\Omega \subseteq \mathfrak{g}_1$ generically. Suppose that $\mu(E', \varphi') = \mu(E, \varphi) - \omega_i^\vee \cdot c$ for some $c \in C$ and $i \in \{1, \ldots, r\}$. Then, (E', φ') is stable.*

Proof. Follows from the fourth condition of Proposition 1.50, as for any Cartan matrix we have that every $b_{jk} \geqslant 0$, and

$$\beta_j(\deg(\mu(E', \varphi'))) = \beta_j(\deg(\mu(E, \varphi))) - \delta_{ij} \leqslant \beta_j(\deg(\mu(E, \varphi))).$$

□

Now we describe another type of interest to us, which is the main focus of study in Chapter 3. When dealing with non-regular Higgs fields, at the opposite end of the spectrum, the first non-trivial type is given by the shortest kind of gradings

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

We call them **Hermitian gradings**. The reason is that they are in correspondence with real forms $G_\mathbb{R} \subseteq G$ of Hermitian type, by which we mean real subgroups with complexification equal to G and such that, given a maximal compact subgroup $H_\mathbb{R} \subseteq G_\mathbb{R}$, the associated symmetric space $G_\mathbb{R}/H_\mathbb{R}$ is Hermitian (i.e. admits an almost complex structure compatible with the Riemannian metric). The geometry of the corresponding Hermitian symmetric space is intricately tied to the algebraic structure of the grading, which can be obtained by considering the Cartan decomposition

$$\mathfrak{g}_\mathbb{R} = \mathfrak{h}_\mathbb{R} \oplus \mathfrak{m}_\mathbb{R}$$

and complexifying it to

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1).$$

For example, the almost complex structure is given by the adjoint action of $-i\zeta$, where ζ is the grading element.

From an algebraic point of view, in order to get a grading with only three pieces it is clear that, up to conjugation, the grading element ζ must correspond to a fundamental coweight $\omega_j^\vee$ such that its pairing with the highest root verifies $\delta(\omega_j^\vee) = 1$, i.e. $\omega_j^\vee$ must be a minuscule dominant coweight (see Section 1.1 for the definition). Not every semisimple Lie algebra possesses these: they are in correspondence with the nodes in the corresponding affine Dynkin diagram which lie in the orbit of the affine root under diagram automorphisms. Table 1.1 records the data for these gradings on each type of simple Lie algebra.

Considering the associated parabolic $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, one immediately sees that they are in correspondence with the maximal parabolic subalgebras of $\mathfrak{g}$ that have abelian nilpotent radical.

Table 1.1 Real forms of $\mathfrak{g}$ of Hermitian type (see [16, Table 1])

Type of $\mathfrak{g}$	ζ	$\mathfrak{g}_0$	$\mathfrak{g}_\mathbb{R}$
A_n	$\omega_j^\vee$	$\mathfrak{s}(\mathfrak{gl}(j,\mathbb{C}) \oplus \mathfrak{gl}(n+1-j,\mathbb{C}))$	$\mathfrak{su}(j, n+1-j)$
B_n	$\omega_1^\vee$	$\mathfrak{so}(2n-1,\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{so}(2, 2n-1)$
C_n	$\omega_n^\vee$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{R})$
D_n	$\omega_1^\vee$	$\mathfrak{so}(2n-2,\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{so}(2, 2n-2)$
	$\omega_{n-1}^\vee, \omega_n^\vee$	$\mathfrak{gl}(n,\mathbb{C})$	$\mathfrak{so}^*(2n)$
E_6	$\omega_1^\vee, \omega_6^\vee$	$\mathfrak{so}(10,\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{e}_{6(-14)}$
E_7	$\omega_7^\vee$	$\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$	$\mathfrak{e}_{7(-25)}$

Definition 1.53 *A fixed point $(E,\varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ is of **Hermitian type** if it reduces to a $(G_0, \mathfrak{g}_1)$-Higgs pair for a Hermitian grading of $\mathfrak{g}$.*

From the point of view of Higgs bundles, there is also a close relation between such fixed points and the associated real form of Hermitian type $G_\mathbb{R}$, namely that, via the non-abelian Hodge correspondence, such Higgs bundles correspond to representations of $\pi_1(C)$ in G whose image lies in the subgroup $G_\mathbb{R} \subseteq G$. Indeed, the locus inside of $\mathcal{M}(G)$ defined by this condition can be obtained as the image of the natural map

$$\mathcal{M}(G_\mathbb{R}) := \mathcal{M}(G_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_1) \to \mathcal{M}(G),$$

where $\mathcal{M}(G_\mathbb{R})$ is called the **moduli space of $G_\mathbb{R}$-Higgs bundles**, and has been extensively studied in the literature e.g. [1, 2, 16, 25, 26, 28, 29]. It plays a particularly important role in the setting of *higher Teichmüller theory*, as they can be used to produce components consisting entirely of discrete and faithful representations in the character variety of $G_\mathbb{R}$.

Remark 1.54 *In general, for any real form $G^\mathbb{R} \subseteq G$ (not necessarily of Hermitian type), the corresponding moduli spaces $\mathcal{M}(G_\mathbb{R}) = \mathcal{M}(H, (\mathfrak{m}_\mathbb{R})_\mathbb{C})$, where H denotes the complexification of $H_\mathbb{R}$, are of great importance. For example, they define holomorphic Lagrangian subvarieties of $\mathcal{M}(G)$ which are crucial in providing examples of BAA-branes within the setting of mirror symmetry explained in Section 1.3. They can be also described as fixed points of involutions in $\mathcal{M}(G)$ [56].*

Example 1.55 *For $G = \mathrm{SL}_n(\mathbb{C})$, recalling the different fixed point types and gradings from Example 1.36, it is clear that fixed points of Hermitian type are those with $k = 2$, i.e. those of type (r_0, r_1), given by a vector bundle*

$$E = E_0 \oplus E_1$$

with $\det(E) \simeq \mathcal{O}_C$ and $\varphi : E_0 \to E_1 \otimes K_C$. They correspond to the real form of Hermitian type $\mathrm{SU}(r_0, r_1) \subseteq \mathrm{SL}_n(\mathbb{C})$. This kind of objects are closely related to the holomorphic triples studied in [27], and also to $SU(r_0, r_1)$-Higgs bundles [26], as explained above.

We conclude this section by introducing a topological invariant that can be attached to a fixed point of the $\mathbb{C}^\times$-action of any type, called the **Toledo invariant**, and recalling its properties. It will be a major tool in the study of Chapter 3. This invariant was originally defined and studied in the setting of representations of $\pi_1(C)$ into a real form $G_\mathbb{R} \subseteq G$ of Hermitian type and, in turn, in the corresponding $G_\mathbb{R}$-Higgs bundle moduli space $\mathcal{M}(G_\mathbb{R})$ as explained above. This version of the Toledo invariant could therefore be computed for fixed points of Hermitian type. However, recent work [14] extends its definition to any moduli space $\mathcal{M}(G_0, \mathfrak{g}_1)$ using the Lie theory behind the Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_1)$, allowing the Toledo invariant to be defined for any kind of fixed point.

While in general there is no analogous real form of Hermitian type inducing this Toledo invariant, there is a larger space of **cyclic Higgs bundles**, corresponding to fixed points by finite order automorphisms of $\mathcal{M}(G)$, which contains $\mathcal{M}(G_0, \mathfrak{g}_1)$ and to which the invariant can be extended retaining many of its properties [50, 58].

We start by fixing a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element $\zeta \in \mathfrak{t}$ (recall that any grading can be conjugated so that the latter holds) and constructing a character of $\mathfrak{g}_0$. Recall that B denotes the Killing form of $\mathfrak{g}$ and B^* its dual form on $\mathfrak{g}^*$.

Definition 1.56 ([14, Definition 3.1]) *The **Toledo character** $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by*

$$\chi_T(x) := B^*(\beta, \beta)B(\zeta, x),$$

where $\beta \in \Delta$ is the longest root such that $\beta(\zeta) = 1$.

The invariance of B guarantees that the result is a character of $\mathfrak{g}_0$, as $B(\zeta, [x, y]) = -B([x, \zeta], y) = 0$ for $x, y \in \mathfrak{g}_0$. The constant factor $B^*(\beta, \beta)$ is a normalisation which allows χ_T to be defined via any invariant bilinear form, getting the same result. The reason why the specific factor that was chosen is $B^*(\beta, \beta)$ is mainly in order to have an agreement with the classical Toledo invariant for representations into real forms of Hermitian type whenever the grading is Hermitian.

Now let E be a principal G_0-bundle. In a similar way to the definition of stability of G-Higgs bundles in Section 1.2, we can define the degree of E associated to χ_T by lifting it to a character of G_0. In general, this requires selecting $q \in \mathbb{Z}_{>0}$ such that $q\chi_T$ lifts to a character $\tilde{\chi} : G_0 \to \mathbb{C}^\times$, and setting

$$\deg(E, \chi_T) := \frac{\deg(E, \tilde{\chi})}{q} = \frac{1}{q} \deg \tilde{\chi}(E).$$

If no such q exists, the degree can still be defined via differential geometric techniques using Chern–Weil theory as explained in Section 1.2 and detailed in [14, Section 4.2].

Definition 1.57 ([14, Definition 5.1]) *Let (E, φ) be a $(G_0, \mathfrak{g}_1)$-Higgs pair. The **Toledo invariant** of (E, φ) is defined to be*

$$\tau(E, \varphi) := \deg(E, \chi_T).$$

The following is an illustrative computation for gradings of $G = \mathrm{SL}_n(\mathbb{C})$ corresponding to the fixed points in Example 1.36, extracted from the author's master thesis [58, Example 21], see also [50, Example 3.11].

Example 1.58 *Consider the fixed points of the $\mathbb{C}^\times$-action for $G = \mathrm{SL}_n(\mathbb{C})$. From Example 1.36, these can be seen as vector bundles $E = E_0 \oplus \cdots \oplus E_{m-1}$ with a Higgs field $\varphi \in H^0(C, \mathrm{End}(E) \otimes K_C)$ such that $\varphi(E_k) \subseteq E_{k+1} \otimes K_C$. We first compute the Toledo character, following the notation of Example 1.36. Recall that in this case the grading element for the Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_1)$ is $\zeta \in \mathfrak{g}_0 = (\bigoplus_{j=0}^{m-1} \mathrm{End}(V_j))_0$ (here the subscript zero denotes the subspace of traceless endomorphisms) given by $\zeta|_{V_j} = (j - \alpha)\,\mathrm{Id}\,|_{V_j}$, where α is the unique constant that makes it traceless. In $\mathrm{SL}_n(\mathbb{C})$ we can choose the invariant form $B(X, Y) = \mathrm{tr}(XY)$ (this is only a scalar multiple of the Killing form, but the Toledo character is independent of this). All the roots of $\mathrm{SL}_n(\mathbb{C})$ have the same length, which is 2 under this invariant form. Thus, we get, for $x = (f_0, \ldots, f_{m-1}) \in \mathfrak{g}_0$, that*

$$\chi_T(x) = B(\zeta, x)B^*(\gamma, \gamma) = 2 \sum_{j=0}^{m-1} (j - \alpha)\,\mathrm{tr}(f_j).$$

A multiple $q\chi_T$ lifts to a character of the group G_0 if $2q(\alpha - j)$ is integral for all $j \in \{0, \ldots, m-1\}$, resulting in

$$\tilde{\chi}(g) = \prod_{j=0}^{m-1} \det(g_j)^{2q(j-\alpha)},$$

for $g = (g_0, \dots, g_{m-1}) \in G_0$. *Such* q *exists as* α *is rational. Then, we have the line bundle*

$$\tilde{\chi}(E) = \bigotimes_{j=0}^{m-1} \det(E)^{\otimes 2q(j-\alpha)},$$

so that

$$\tau(E, \varphi) = 2 \sum_{j=0}^{m-1} (j - \alpha) \deg E_j.$$

The value of α, *if explicitly computed, results in* $\alpha = \frac{\sum_{j=0}^{m-1} j r_j}{\sum_{j=0}^{m-1} r_j}$ *for* $r_j = \operatorname{rank} E_j$, *so we see that the resulting invariant depends on the degrees and ranks of each piece* E_j.

The case of $m = 2$ *corresponds the Hermitian grading in Example 1.55. If we let* (p, q) *and* (a, b) *be the ranks and degrees, respectively, of each of the two pieces* E_0 *and* E_1, *the previous formula reads*

$$\tau(E, \varphi) = 2\frac{pb - qa}{p + q},$$

which is the Toledo invariant for $\mathrm{SU}(p, q)$*-Higgs bundles defined in [26]. Notice that working with* $\mathrm{SU}(p, q)$ *(instead of* $\mathrm{U}(p, q)$*) imposes the additional restriction* $\deg(E_0) = -\deg(E_1)$, *so we can substitute* $b = -a$ *in order to obtain a simplified expression for the invariant. Similarly, for larger values of* m *we can use the constraint* $\sum_{j=0}^{m-1} \deg E_j = 0$ *to obtain an expression with one less parameter.*

One of the main features of the Toledo invariant is the fact that the stability of the fixed point (E, φ) enforces bounds on its value. In the original setting of Hermitian representations and their corresponding Higgs bundles, this is known as the **Milnor–Wood inequality**, whereas in the setting of arbitrary fixed points of the $\mathbb{C}^\times$-action in $\mathcal{M}(G)$ it was termed **Arakelov–Milnor inequality**.

In order to state these bounds, we need to introduce more representation theory of Vinberg $\mathbb{C}^\times$-pairs. First, Vinberg proved ([126], see also [85, Theorem 10.19]) that the representation $(G_0, \mathfrak{g}_1)$ has finitely many orbits, therefore there is a unique dense open orbit $\Omega \subseteq \mathfrak{g}_1$ (notice that this generalises Remark 1.38). Representations with such an orbit are called **prehomogeneous vector spaces**.

Given a Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_1)$ and any element $e \in \mathfrak{g}_1$, we may complete it to an $\mathfrak{sl}_2(\mathbb{C})$-triple (e, h, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$, see e.g. [85, Lemma 10.18].

Definition 1.59 *The* ***Toledo rank*** *of* $e \in \mathfrak{g}_1$ *is defined to be*

$$\mathrm{rk}_T(e) := \frac{\chi_T(h)}{2}.$$

The ***Toledo rank*** *of* $(G_0, \mathfrak{g}_1)$ *is defined by*

$$\mathrm{rk}_T(G_0, \mathfrak{g}_1) := \mathrm{rk}_T(e_\Omega),$$

for any $e_\Omega \in \Omega \subseteq \mathfrak{g}_1$.

Given a $(G_0, \mathfrak{g}_1)$*-Higgs pair* (E, φ), *the* ***Toledo rank*** *of* φ *is defined by*

$$\mathrm{rk}_T(\varphi) := \mathrm{rk}_T(e_\varphi),$$

for any $e_\varphi \in \Omega_\varphi \subseteq \mathfrak{g}_1$, *where* Ω_φ *is the orbit taken generically by* φ *(which exists because there are finitely many orbits).*

We remark that the inequalities $0 \leqslant \mathrm{rk}_T(e) \leqslant \mathrm{rk}_T(G_0, \mathfrak{g}_1)$ always hold [14, Proposition 3.13]. For fixed points of Hermitian type in $G = \mathrm{SL}_n(\mathbb{C})$, where $\varphi : E_0 \to E_1 \otimes K_C$ pointwise is nothing but a matrix, the Toledo rank agrees with the generic rank of the matrix in the usual sense.

Theorem 1.60 (Arakelov–Milnor inequality [14, Theorem 5.3]). *Let $(E, \varphi) \in \mathcal{M}(G_0, \mathfrak{g}_1)$. Then, the following inequalities hold.*

$$-\mathrm{rk}_T(\varphi)(2g-2) \leqslant \tau(E, \varphi) \leqslant 0.$$

In the source cited above the inequality also accounts for different notions of stability parametrised by a real number, but we will not need them in the present work.

We conclude by explaining another of the main features of the Toledo invariant, which is the **Cayley correspondence**. In this context, it refers to a rigidity phenomenon that occurs whenever we look at components of $\mathcal{M}(G_0, \mathfrak{g}_1)$ whose Toledo invariant attains the bound

$$\tau = -\tau_{\max} = -\mathrm{rank}_T(G_0, \mathfrak{g}_1)(2g-2).$$

These are called **maximal** components. It turns out that they are connected and easily described by the Cayley correspondence. First we need to define the following notion.

Definition 1.61 *A Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_1)$ is said to be **JM-regular** if there exists an $\mathfrak{sl}_2(\mathbb{C})$-triple $(e, 2\zeta, f)$ with $e \in \Omega \in \mathfrak{g}_1$ and $f \in \mathfrak{g}_{-1}$.*

In other words, JM-regular Vinberg pairs are those associated to the grading given by $\mathrm{ad}(h)/2$, where (e, h, f) is a $\mathfrak{sl}_2(\mathbb{C})$-triple in $\mathfrak{g}$ such that the eigenvalues of $\mathrm{ad}(h)$ are even. The existence of this $\mathfrak{sl}_2(\mathbb{C})$-triple provides additional tools in the study of these representations.

Remark 1.62 *In the case of Hermitian gradings $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the notion of JM-regularity is equivalent to the associated Hermitian symmetric space $G_{\mathbb{R}}/H_{\mathbb{R}}$ being of **tube type**. This means that it admits a realisation as a bounded domain (due to Harish-Chandra) which is biholomorphic to a generalised upper half plane, see [88].*

We conclude by stating the Cayley correspondence for JM-regular Vinberg $\mathbb{C}^\times$-pairs.

Theorem 1.63 (Cayley correspondence for $(G_0, \mathfrak{g}_1)$-Higgs pairs in the JM-regular case ([14, Theorem 6.2])). *Let $(G_0, \mathfrak{g}_1)$ be a Vinberg $\mathbb{C}^\times$-pair and $\mathcal{M}^{\max}(G_0, \mathfrak{g}_1) \subseteq \mathcal{M}(G_0, \mathfrak{g}_1)$ denote the locus of components with maximal Toledo invariant $\tau = -\tau_{\max}$. Then, there is an isomorphism of algebraic varieties*

$$\mathcal{M}^{\max}(G_0, \mathfrak{g}_1) \simeq \mathcal{N}(G_0^e),$$

where $\mathcal{N}(G_0^e)$ is the moduli space of isomorphism classes of polystable principal bundles over C with structure group the stabiliser $G_0^e \subseteq G_0$ of $e \in \Omega \subseteq \mathfrak{g}_1$.

The previous isomorphism is given as follows. For a JM-regular Vinberg $\mathbb{C}^\times$-pair, the equality $\tau = -\tau_{\max}$ is equivalent to $\varphi \in H^0(E(\Omega) \otimes K_C)$, that is, it is equivalent to the Higgs field taking values in the open dense orbit at every point of C. Therefore, one can construct a suitable twist $E \otimes E_T^{-1}$, where E_T is a uniformising T-bundle defined by the property $E_T(\langle e \rangle) \simeq K_C^{-1}$, in such a way that $H^0(E(\Omega) \otimes K_C) = H^0((E \otimes E_T^{-1})(\Omega)) = H^0((E \otimes E_T^{-1})(G_0/G_0^e))$. We then see that the Higgs field determines a reduction of structure group to G_0^e, which is the corresponding polystable G_0^e-bundle $E' \in \mathcal{N}(G_0^e)$. Conversely, E' allows to recover (E, φ) by setting $E = (E' \otimes E_T)(G_0)$ and $\varphi = e$.

When the Vinberg pair is not JM-regular, there are no stable Higgs pairs in $\mathcal{M}^{\max}(G_0, \mathfrak{g}_1)$: they are all polystable, since they always admit degree zero reductions to a *subpair* $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ which is JM-regular. This is the content of [14, Proposition 6.15, Theorem 6.16]. Interestingly, however, in this case there exist pairs where the Higgs field takes values everywhere in the open dense orbit even when $\tau \neq -\tau_{\max}$.

1.5 Upward flows in $\mathcal{M}(G)$ and very stable Higgs bundles

In this section we introduce the main ingredients of the Białynicki-Birula partition of a $\mathbb{C}^\times$-action on a semiprojective variety and apply them to the context of $\mathcal{M}(G)$ in order to define our main object of study: very stable Higgs bundles.

Definition 1.64 *A normal complex quasiprojective variety X equipped with a $\mathbb{C}^\times$-action is called* ***semiprojective*** *if the fixed point locus $X^{\mathbb{C}^\times}$ is projective and for every $x \in X$ the limit $\lim_{\lambda\to 0} \lambda x$ exists.*

In the algebraic setting, the existence of limit means that there exists a $\mathbb{C}^\times$-equivariant morphism $f : \mathbb{A}^1_{\mathbb{C}} \to X$ mapping $\lambda \in \mathbb{C}^\times \subseteq \mathbb{A}^1_{\mathbb{C}}$ to λx. Then, the limit is defined to be the image of 0. Similarly we can define limits at ∞ by inverting the $\mathbb{C}^\times$-action on X. Given a semiprojective variety, we obtain a decomposition by using the limit points of the $\mathbb{C}^\times$-action.

Definition 1.65 *Let X be a semiprojective variety and $\alpha \in X^{\mathbb{C}^\times}$. The* ***upward flow from*** *α is*

$$W_\alpha^+ := \{x \in X : \lim_{\lambda\to 0} \lambda x = \alpha\} \subseteq X.$$

The upward flows define the ***Białynicki-Birula partition***

$$X = \bigsqcup_{\alpha \in X^{\mathbb{C}^\times}} W_\alpha^+.$$

Similarly, the ***downward flow from*** *α is*

$$W_\alpha^- := \{x \in X : \lim_{\lambda\to \infty} \lambda x = \alpha\} \subseteq X,$$

and the ***core*** *of X is defined to be*

$$\mathcal{C} := \bigsqcup_{\alpha \in X^{\mathbb{C}^\times}} W_\alpha^- \subseteq X.$$

Given a smooth fixed point $\alpha \in X^{s\mathbb{C}^\times}$, its upward and downward flows can be described by studying the tangent space $T_\alpha X$. Since α is fixed, $T_\alpha X$ has an induced $\mathbb{C}^\times$-action which provides a weight space decomposition

$$T_\alpha X = \bigoplus_{k\in\mathbb{Z}} (T_\alpha X)_k.$$

Let $T_\alpha X^+ := \bigoplus_{k>0} (T_\alpha X)_k$ be the subspace of positive weights and $T_\alpha X^- := \bigoplus_{k<0} (T_\alpha X)_k$ the subspace of negative weights.

Proposition 1.66 ([11], [70, Proposition 2.1]). *Let X be a semiprojective variety and $\alpha \in X^{s\mathbb{C}^\times}$ a smooth fixed point. The upward flow W_α^+ (resp. the downward flow W_α^-) is a locally closed $\mathbb{C}^\times$-invariant subvariety of X isomorphic to $T_\alpha^+ X$ (resp. $T_\alpha^- X$) as $\mathbb{C}^\times$-varieties.*

We now recall the definition [70, Definition 2.12] of a very stable point in a semiprojective variety.

Definition 1.67 *A smooth fixed point $\alpha \in X^{s\mathbb{C}^\times}$ is **very stable** if $W_\alpha^+ \cap \mathcal{C} = \{\alpha\}$.*

We will use the following equivalent characterisation given in [70, Proposition 2.14].

Proposition 1.68. *The point $\alpha \in X^{s\mathbb{C}^\times}$ is very stable if and only if $W_\alpha^+ \subseteq X$ is closed.*

Lastly, we consider semiprojective varieties with the additional structure of a symplectic form $\omega \in H^0(C, \Omega_X^2)$ of weight one, that is, such that $\Phi_\lambda^* \omega = \lambda\omega$ for $\lambda \in \mathbb{C}^\times$, where $\Phi_\lambda : X \to X$ is given by the $\mathbb{C}^\times$-action on X. In this case, the upward flows are Lagrangian subvarieties.

Proposition 1.69 ([70, Proposition 2.10]). *In the above setting, the upward flows W_α^+ for $\alpha \in X^{s\mathbb{C}^\times}$ are Lagrangian subvarieties of X.*

Now we may apply the above to $\mathcal{M}(G)$ with its $\mathbb{C}^\times$-action described in Section 1.4. The $\mathbb{C}^\times$-equivariance of h_G, together with its properness, imply that $\mathcal{M}(G)$ is a semiprojective variety. Indeed, as $0 \in \mathcal{A}(G)$ is the only fixed point in the base, the fixed point locus $\mathcal{M}(G)^{\mathbb{C}^\times}$ has to be a closed subvariety of $h_G^{-1}(0)$, so it is projective. Furthermore, given $(E, \varphi) \in \mathcal{M}(G)$, the limit of $h_G(E, \lambda\varphi)$ when $\lambda \to 0$ exists and equals $0 \in \mathcal{A}(G)$, so that properness of h_G implies the existence and uniqueness of the limit of $(E, \lambda\varphi)$ when $\lambda \to 0$.

Moreover, a similar argument shows that the limit of $(E, \lambda\varphi)$ when $\lambda \to \infty$ exists if and only if $(E, \varphi) \in h_G^{-1}(0)$. In other words, there is an agreement $\mathcal{C} = h_G^{-1}(0)$ between the core and the fibre over 0 of h_G, sometimes referred to as the **nilpotent cone**. Therefore, we have the following definition.

Definition 1.70 ([70, Definitions 2.12 and 4.1]) *A smooth fixed point $(E, \varphi) \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ is **very stable** if the only point in both its upward flow and the nilpotent cone is itself:*

$$W^+_{(E,\varphi)} \cap h_G^{-1}(0) = \{(E, \varphi)\}.$$

By Proposition 1.68, a smooth fixed point (E, φ) is very stable if and only if its upward flow $W^+_{(E,\varphi)}$ is closed. Moreover, by the $\mathbb{C}^\times$-equivariance of the Hitchin map h_G and the fact that $h_G^{-1}(0)$ consists of the nilpotent polystable G-Higgs bundles, a smooth fixed point is very stable if and only if the only nilpotent Higgs bundle on its upward flow is the fixed point itself. Lastly, [70, Lemma 4.6] shows that (E, φ) is very stable if and only if the restriction of the Hitchin map

$$h_G|_{W^+_{(E,\varphi)}} : W^+_{(E,\varphi)} \to \mathcal{A}(G)$$

is a finite flat surjective morphism.

Furthermore, recall from Section 1.4 that the $\mathbb{C}^\times$-action scales the holomorphic symplectic form ω with weight one. As a consequence, Proposition 1.69 applies and the upward flows define holomorphic Lagrangian subvarieties. Therefore, they provide examples of supports for BAA-branes of special interest to mirror symmetry, as explained in Section 1.3. Very stable upward flows correspond to closed holomorphic Lagrangian subvarieties, which have the simplest geometry (they are isomorphic to affine spaces, see Proposition 1.66) and the simplest intersection with the Hitchin fibres, as remarked above, making them especially relevant in the study of mirror symmetry. Moreover, the study of very stable Higgs bundles ties into the geometry and topology of the nilpotent cone $h_G(0)^{-1}$ since they correspond to the closed Białynicki–Birula cells for the $\mathbb{C}^\times$-action on the projective variety $h_G(0)^{-1}$.

Now we describe the upward and downward flows for any smooth fixed point $(E,\varphi)\in\mathcal{M}(G)^{s\mathbb{C}^\times}$. Recall from Proposition 1.66 that this amounts to studying the weights of the $\mathbb{C}^\times$-action induced on the tangent space at a fixed point. We will do so algebraically via the deformation theory of Proposition 1.19, however there is an analytic proof [34, Proposition 4.2] in the $G=\mathrm{GL}_n(\mathbb{C})$ case which can be readily adapted to the case of arbitrary semisimple structure group G. An algebraic proof also exists in the $G=\mathrm{GL}_n(\mathbb{C})$ case [70, Proposition 3.4].

We start by considering the $\mathbb{Z}$-grading $\mathfrak{g}:=\bigoplus_j\mathfrak{g}_j$ such that (E,φ) reduces to a $(G_0,\mathfrak{g}_k)$-Higgs pair. Let $\zeta\in\mathfrak{g}_0$ be its grading element. As explained in Section 1.4, without loss of generality we may assume that $k=1$, otherwise we can use the subalgebra $\hat{\mathfrak{g}}=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_{jk}$ and rescale the grading.

Since ζ is semisimple with integral eigenvalues, it exponentiates to a cocharacter

$$\xi_\bullet:\mathbb{C}^\times\to G/Z(G)\simeq G_{\mathrm{ad}}$$

of the adjoint group, satisfying $\mathrm{Ad}(\xi_{\exp(t)})=\exp(\mathrm{ad}(t\zeta))$ for $t\in\mathbb{C}$. Let $\mathfrak{p}_{\leqslant 0}:=\bigoplus_{j\leqslant 0}\mathfrak{g}_j\subseteq\mathfrak{g}$, which is a parabolic subalgebra of $\mathfrak{g}$, and $P_{\leqslant 0}\subseteq G$ the corresponding parabolic subgroup. Let $\mathfrak{n}_{<0}:=\bigoplus_{j<0}\mathfrak{g}_j\subseteq\mathfrak{g}$ be the nilpotent radical and $U_{<0}\subseteq P_{\leqslant 0}$ the unipotent radical. We have the Levi decomposition $P_{\leqslant 0}\simeq U_{<0}\rtimes G_0$. Note that $P_{\leqslant 0}$ acts on $\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1$.

Now suppose that (E,φ) is a G-Higgs bundle which admits a reduction of structure group $\sigma\in H^0(C,E(G/P_{\leqslant 0}))$ to a $(P_{\leqslant 0},\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)$-Higgs pair. For such an object we may define a G-Higgs bundle which reduces to a $(G_0,\mathfrak{g}_1)$-Higgs pair, known as its **associated graded** and denoted by $\mathrm{Gr}_\sigma(E,\varphi)=(\mathrm{Gr}_\sigma(E),\mathrm{Gr}_\sigma(\varphi))$, as follows. The principal G_0-bundle is simply the quotient

$$\mathrm{Gr}_\sigma(E):=E_\sigma/U_{<0}\simeq E_\sigma(P_{\leqslant 0}/U_{<0}),$$

which has the structure of a principal $P_{\leqslant 0}/U_{<0}$-bundle, and we have $P_{\leqslant 0}/U_{<0}\simeq G_0$. For the Higgs field, consider the $P_{\leqslant 0}$-equivariant quotient map

$$\pi_0:\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1\to(\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)/\mathfrak{p}_{\leqslant 0}\simeq\mathfrak{g}_1.$$

Since $U_{<0}$ acts trivially on $(\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)/\mathfrak{p}_{\leqslant 0}$, we have that $E_\sigma((\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)/\mathfrak{p}_{\leqslant 0})\simeq\mathrm{Gr}_\sigma(E)((\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)/\mathfrak{p}_{\leqslant 0})\simeq\mathrm{Gr}_\sigma(E)(\mathfrak{g}_1)$. Therefore, π_0 induces a map

$$H^0(C,E_\sigma(\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)\otimes K_C)\to H^0(C,\mathrm{Gr}_\sigma(E)(\mathfrak{g}_1)\otimes K_C)$$

and $\mathrm{Gr}_\sigma(\varphi)$ is defined to be the image of φ via this map.

Proposition 1.71. *Let (E,φ) be a smooth (stable and simple) $\mathbb{C}^\times$-fixed point. Then, a Higgs bundle $(E',\varphi')\in\mathcal{M}(G)$ is contained the upward flow $W^+_{(E,\varphi)}\subseteq\mathcal{M}(G)$ if and only if it admits a reduction $\sigma\in H^0(C,E'(G/P_{\leqslant 0}))$ to a $(P_{\leqslant 0},\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1)$-Higgs pair such that*

$$\mathrm{Gr}_\sigma(E',\varphi')\simeq(E,\varphi).$$

Proof. Suppose first that such reduction exists. This reduction induces (by trivialising E_σ) a trivialisation $\{(C_i,t_i)\}_{i\in I}$ of E, where each $C_i\subseteq C$ is an open subset and $t_i:E'|_{C_i}\xrightarrow{\sim}C_i\times G$, in such a way that the transition functions $g_{ij}:C_i\cap C_j\to G$ given by considering $t_i\circ t_j^{-1}$ have image contained in $P_{\leqslant 0}$, and the local expressions $\varphi'_i:C_i\to\mathfrak{g}$ of the Higgs field (obtained by also trivialising K_C over each C_i) have image in $\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1$. For $k\leqslant 1$, let $\varphi'_{i,k}:C_i\to\mathfrak{g}_k$ denote the component of φ'_i in $\mathfrak{g}_k$. The local expressions of $\lambda\varphi'$ are given by $\lambda\varphi'_i$.

Now we choose an alternative trivialisation, given by replacing each t_i with $t_i^\lambda:=L_{\xi_\lambda^{-1}}\circ t_i$ where $L_{\xi_\lambda^{-1}}$ is the automorphism of the trivial bundle given by left multiplication by ξ_λ^{-1}. Thus, this new trivialisation has transition functions $g_{ij}^\lambda=\xi_\lambda^{-1}g_{ij}\xi_\lambda$. Moreover, the local expressions of $\lambda\varphi'$ become

$$\lambda\varphi_i'^{\lambda} := \mathrm{Ad}_{\xi_\lambda^{-1}}(\lambda\varphi_i') = \sum_{k\leqslant 1}\lambda^{1-k}\varphi'_{i,k}.$$

Using these trivialisations we see that there is a limiting point of this family when $\lambda \to 0$: the bundle is given by the cocycle $g_{ij}^0 := \lim_{\lambda\to 0}\xi_\lambda^{-1}g_{ij}\xi_\lambda$, which exists and takes values in G_0 because g_{ij} takes values in $P_{\leqslant 0}$ (it is given by applying the projection to the Levi factor G_0 after g_{ij}), and the Higgs field is given by local expressions

$$\varphi_i'^{0} := \lim_{\lambda\to 0}\lambda\varphi_i'^{\lambda} = \lim_{\lambda\to 0}\sum_{k\leqslant 1}\lambda^{1-k}\varphi'_{i,k} = \varphi'_{i,1}.$$

This is precisely the local data defining $\mathrm{Gr}_\sigma(E',\varphi')$, which by assumption is isomorphic to (E,φ) which is stable. Therefore, $\lim_{\lambda\to 0}(E',\lambda\varphi') = (E,\varphi)$ in $\mathcal{M}(G)$.

Conversely, we now prove that every (E',φ') in the upward flow of (E,φ) is endowed with such a reduction. For this, recall from Proposition 1.66 that, since $(E,\varphi)\in\mathcal{M}(G)$ is a smooth point, the upward flow is given by the subspace of positive weights of the tangent space to $\mathcal{M}(G)$ at (E,φ). From Proposition 1.19, the tangent space is obtained as the first hypercohomology of the complex

$$C^\bullet : E(\mathfrak{g}) \xrightarrow{[\varphi,-]} E(\mathfrak{g})\otimes K_C.$$

Notice that $\varphi\in H^0(E(\mathfrak{g}_1)\otimes K_C)$, the complex splits as the direct sum of the following complexes

$$C_j^\bullet : E(\mathfrak{g}_j) \xrightarrow{[\varphi,-]} E(\mathfrak{g}_{j+1})\otimes K_C,$$

so that the tangent space splits naturally as

$$T_{(E,\varphi)}\mathcal{M}(G) = \bigoplus_{j\in\mathbb{Z}}\mathbb{H}^1(C_j^\bullet).$$

Now we claim that the induced $\mathbb{C}^\times$-action on $T_{(E,\varphi)}\mathcal{M}(G)$ has weight $-j$ on $\mathbb{H}^1(C_j^\bullet)$. Indeed, consider a deformation $(\tilde{E},\tilde{\varphi})\in\mathbb{H}^1(C_j^\bullet)$. This means that locally, for a deformation parameter $\varepsilon\in\mathbb{C}[\varepsilon]/\varepsilon^2$, we have $\tilde{\varphi} = \varphi + \varepsilon\varphi_{j+1}$ where φ_{j+1} is a local section of $E(\mathfrak{g}_{j+1})\otimes K_C$ obtained from the Čech cocycle defining the hypercohomology class. Moreover, $\tilde{E}$ is obtained by glueing local copies of the trivial deformation of E via the isomorphisms $1+\varepsilon s$ where the s are local sections of $E(\mathfrak{g}_j)$, also obtained from the cocycle. The $\mathbb{C}^\times$-action sends $(\tilde{E},\tilde{\varphi})\mapsto(\tilde{E},\lambda\tilde{\varphi}) = (\tilde{E},\lambda\varphi+\varepsilon\lambda\varphi_{j+1})$. After applying the automorphism given by ξ_λ (acting as $\mathrm{Ad}_{\xi_\lambda^{-1}}$ on the Lie algebra), the resulting bundle is $(\tilde{E}',\varphi+\varepsilon\lambda^{-j}\varphi_{j+1})$ where $\tilde{E}'$ is the deformation obtained from the sections $\lambda^{-j}s$ since they belong to $E(\mathfrak{g}_j)$. Thus, the action of $\lambda\in\mathbb{C}^\times$ on the tangent space is

$$(\{s\},\{\varphi_{j+1}\})\mapsto(\{\lambda^{-j}s\},\{\lambda^{-j}\varphi_{j+1}\}),$$

as claimed.

Therefore, the upward flow is obtained from the deformations in $\bigoplus_{j<0}\mathbb{H}^1(C_j^\bullet)$, which are always given by a $P_{\leqslant 0}$-bundle (because E reduces structure group to G_0 and we only deform using sections of $E(\mathfrak{n}_{<0})$) with Higgs field taking values in $\mathfrak{p}_{\leqslant 0}\oplus\mathfrak{g}_1$ (because φ is deformed by adding local sections of $E(\mathfrak{p}_{\leqslant 0})\otimes K_C$). □

Example 1.72 *In the $G=\mathrm{SL}_n(\mathbb{C})$ case of Example 1.36, where the grading arises from a splitting $\mathbb{C}^n = V_0\oplus\cdots\oplus V_{k-1}$, we have that*

$$P_{\leqslant 0} = \{A \in \mathrm{SL}_n(\mathbb{C}) : A(V_j) \subseteq \bigoplus_{i \leqslant j} V_i\},$$

in other words, $P_{\leqslant 0}$ consists of the automorphisms of $\mathbb{C}^n$ that preserve the filtration

$$0 \subseteq V_0 \subseteq V_0 \oplus V_1 \subseteq \cdots \subseteq \mathbb{C}^n,$$

and $\mathfrak{p}_{\leqslant 0} \oplus \mathfrak{g}_1$ are the endomorphisms of degree one with respect to the filtration. Therefore, the upward flow of a fixed point $(E = E_0 \oplus \cdots \oplus E_{k-1}, \varphi)$ where $\varphi(E_j) \subseteq E_{j+1} \otimes K_C$ is given by the Higgs bundles (E', φ') admitting a filtration by subbundles

$$0 \subseteq E'_0 \subseteq E'_1 \subseteq \cdots \subseteq E'_{k-1} = E',$$

with $\varphi'(E'_j) \subseteq E'_{j+1} \otimes K_C$ and such that the associated graded $(\bigoplus_j E'_j/E'_{j-1}, \overline{\varphi'})$ is isomorphic to (E, φ). This is [70, Proposition 3.4].

In a completely analogous manner (but changing the signs appropriately) the next proposition follows.

Proposition 1.73. *Let (E, φ) be a smooth (stable and simple) $\mathbb{C}^\times$-fixed point corresponding to a $\mathbb{Z}$-grading of $\mathfrak{g}$ with parabolic $\mathfrak{p} = \bigoplus_{j \geqslant 0} \mathfrak{g}_j$. Let $P \subseteq G$ be the corresponding parabolic subgroup. Denote by $\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}_j$ its nilradical, and by $U \subseteq P$ the unipotent radical. Then, the downward flow of (E, φ) is given by the Higgs bundles $(E', \varphi') \in \mathcal{M}(G)$ that have a reduction $\sigma \in H^0(C, E'(G/P))$ to $(P, \mathfrak{g}_{\geqslant 1})$-Higgs pairs such that*

$$\mathrm{Gr}_\sigma(E', \varphi') \simeq (E, \varphi).$$

Note that, in the previous statement, $\mathrm{Gr}_\sigma(E', \varphi')$ is defined analogously by setting $\mathrm{Gr}_\sigma(E') := E'_\sigma/U$ and projecting φ' to its component in $\mathfrak{g}_1$.

We conclude this section by constructing upward flows of Borel type satisfying $\mu(E, \varphi) = 1$, where μ denotes the multiplicity divisor from Definition 1.45. As noted in Remark 1.47, there is a unique such upward flow up to tensoring by a $Z(G)$-bundle. A key aspect of these upward flows is that they are sections of the Hitchin map h_G. In particular, the well-known Hitchin sections from [77] are examples of such upward flows, and the construction follows closely that of Hitchin, but whenever G is not of adjoint type there are generally more upward flows with $\mu(E, \varphi) = 1$ than Hitchin sections.

Example 1.74 *Consider a smooth fixed point $(E^0, \varphi^0) \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ of Borel type that satisfies $\mu(E^0, \varphi^0) = 1$, i.e. its Higgs field φ^0 is everywhere regular. Denote by E_T the corresponding reduction of structure group of E^0 to $T = G_0$. This implies that all the sections $\pi_j(\varphi^0) \in H^0(E_T(\mathfrak{g}_{\beta_j}) \otimes K_C)$ are nowhere vanishing. Therefore, the line bundles $E_T(\mathfrak{g}_{\beta_j})$ are all isomorphic to K_C^{-1}. Whenever $G = G_{\mathrm{ad}}$ is of adjoint type, the previous condition determines the T-bundle E_T uniquely. In general, there are $|Z(G)|^{2g}$ possible choices for such an E_T.*

Moreover, since all the line bundles $E_T(\mathfrak{g}_{\beta_j})$ are isomorphic, this E_T admits a reduction of structure group $E_{T'}$ to $T' \subseteq T$, the one-dimensional subtorus spanned by the principal cocharacter corresponding to the grading element $\zeta = \sum_{j=1}^r \omega_j^\vee$. Consider the regular nilpotent element $e := \sum_{j=1}^r X_{\beta_j} \in \mathfrak{g}_1$ and the associated principal $\mathfrak{sl}_2$-triple $(2\zeta, e, f)$ obtained by the Jacobson–Morozov theorem.

By construction, since $[\zeta, e] = e$, we have $E_{T'}(\langle e \rangle) \simeq K_C^{-1}$. Therefore, we have a constant section $e \in H^0(E_{T'}(\langle e \rangle) \otimes K_C)$ and, extending the structure group of the pair $(E_{T'}, e)$ to G, we get a K_C-twisted G-Higgs bundle (E^0, φ^0) which, by construction, reduces to a K_C-twisted $(T, \mathfrak{g}_1)$-Higgs pair (in fact, to a $(T', \langle e \rangle)$-Higgs pair).

Now we explicitly construct its upward flow as a section of the Hitchin map, as was done originally in [77]. Given a point $a = (a_1, \dots, a_r) \in \mathcal{A}(G) = \bigoplus_{i=1}^r H^0(C, K_C^{\deg(p_i)})$ *in the Hitchin base, decompose* $\mathfrak{g}$ *into irreducible representations of the* $\mathfrak{sl}_2(\mathbb{C})$*-representation induced by* $(2\zeta, e, f)$*, getting*

$$\mathfrak{g} = \bigoplus_{i=1}^{r} V_i.$$

There are $r = \operatorname{rank}\mathfrak{g}$ *summands, and they can be ordered so that* $\dim V_i = 2\deg p_i - 1$ *where the* $p_i \in \mathbb{C}[\mathfrak{g}]^G$ *are the chosen basis elements of invariant polynomials for the construction of* h_G*. Thus, if* $f_i \in V_i$ *is a lowest weight vector, we have that* $[2\zeta, f_i] = (-2\deg p_i + 2)f_i$*, and hence* $E_{T'}(\langle f_i \rangle) \otimes K_C \simeq K_C^{\deg p_i - 1} \otimes K_C \simeq K_C^{\deg p_i}$*. This means that we have a well-defined constant section* $f_i \in H^0(C, E_{T'}(\langle f_i \rangle) \otimes K_C \otimes K_C^{-\deg p_i})$ *and hence* $a_i f_i \in H^0(C, E^0(\mathfrak{g}) \otimes K_C)$*. This results in a well defined Higgs field*

$$\varphi^a := \varphi^0 + \sum_{i=1}^{r} a_i f_i.$$

The f_i *can be selected so that* $h_G(E^0, \varphi^a) = a$*. We also have that* (E^0, φ^a) *are smooth points [77, Section 5] in* $\mathcal{M}(G)$*. This also follows directly from Propositions 1.49 and 1.50, since it is enough to check smoothness of the special point* (E^0, φ^0) *and we have* $\deg(\mu(E^0, \varphi^0)) = 0$*. By the* $\mathbb{C}^\times$*-equivariance of* h_G*, we have the subvariety*

$$\mathcal{A}(G) \simeq \{(E^0, \varphi^a) : a \in \mathcal{A}(G)\} \subseteq W^+_{(E^0, \varphi^0)}.$$

However, both sides are vector spaces of the same dimension, hence equality holds. It then follows from this explicit description of the upward flow that (E^0, φ^0) *is very stable. When* $G = G_{\mathrm{ad}}$ *is of adjoint type, there is a unique such fixed point usually referred to as* ***canonical uniformising Higgs bundle****. For other choices of groups, this construction includes the* ***uniformising Higgs bundles*** *from [77] or [14, Example 4.4] (for the Borel grading), but generally more as* $|Z(G)|$ *is typically greater than* 2*.*

1.6 Moduli spaces of parabolic *G*-Higgs bundles

We start by defining parabolic bundles over the punctured curve (C, D), where C is a smooth projective complex curve of genus g with canonical line bundle K_C and $D = c_1 + \cdots + c_s$ is a reduced effective divisor marking s punctures in C. Note that in the parabolic setting we allow any genus g, the only relevant condition to get reasonable moduli spaces is that $\deg K_C(D) > 0$.

Let $\mathfrak{t}_{\mathbb{R}} \subseteq \mathfrak{t}$ denote the $\mathbb{R}$-span of $P_{\Delta^\vee}$ (in other words, it is the Cartan subalgebra for the split real form of $\mathfrak{g}$). At each puncture c_i, we choose **parabolic weights** $\alpha_i \in \mathfrak{t}_{\mathbb{R}}$, and denote by $\alpha := (\alpha_1, \dots, \alpha_s)$ the vector of parabolic weights.

For simplicity in the definitions, we **assume** that the parabolic weights lie in the interior of the **standard Weyl alcove** of $\mathfrak{t}$. This means that, for each α_i, we assume:

- $\beta_j(\alpha_i) > 0$, for every $\beta_j \in \Pi$.
- $\delta(\alpha_i) < 1$, recalling that $\delta \in \Delta$ denotes the highest root.

Remark 1.75 *It is possible to consider more general weights, in the closure of the standard Weyl alcove (which, in the context of non-abelian Hodge theory, covers all the possible conjugacy classes of the compact part of the monodromy, see [15, Appendix A]) or even in any alcove of* $\mathfrak{t}_{\mathbb{R}}$*. This introduces more subtleties in the definition of parabolic G-Higgs bundles [15, Sections 2–4].*

The alternative, which is more convenient for our purposes, would be to work with ***parahoric*** *torsors [5] and* ***logahoric*** *Higgs torsors [91] for the corresponding weight. For our assumption of weights in the interior of the standard alcove, this notion is equivalent to the parabolic one presented in this paper (see [15, Section 3]).*

Definition 1.76 *A* ***parabolic*** G***-bundle*** *over* (C, D) *with weights* $\alpha = (\alpha_1, \dots, \alpha_s)$ *is a pair* (E, Q) *where*

- E *is a principal* G*-bundle over* C*.*
- $Q = (Q_1, \dots, Q_s)$ *is a choice of a* ***parabolic structure*** *for each* $c_i \in D$*, that is, a* B*-orbit* $Q_i \in E|_{c_i}/B$*.*

The choice of a parabolic structure at c_i is equivalent to choosing a subgroup $Q_i' \subseteq E(G)|_{c_i}$ isomorphic to B in the fibre of the adjoint bundle $E(G)$. Indeed, said fibre can be described as the space of G-equivariant maps

$$f : E|_{c_i} \to G$$

so given a B-orbit $eB \in E|_{c_i}/B$ we get a subgroup

$$Q_i' = \{f \in E(G)|_{c_i} : f(e) \in B\}.$$

Conversely, given such a subgroup, there is an element $e \in E|_{c_i}$ such that $f(e) \in B$ for all $f \in Q_i'$, determining the orbit $eB \in E|_{c_i}$ (which is well-defined as B is self-normalising).

Therefore, we get an induced Borel subalgebra $\mathfrak{q}_i \subseteq E(\mathfrak{g})|_{c_i}$. Let $\mathfrak{n}_i \subseteq \mathfrak{q}_i$ denote its nilpotent radical. Define the subsheaves $SPE(\mathfrak{g}) \subseteq PE(\mathfrak{g}) \subseteq E(\mathfrak{g})$ via the exact sequences

$$0 \to PE(\mathfrak{g}) \to E(\mathfrak{g}) \to \bigoplus_{i=1}^{s} E(\mathfrak{g})|_{c_i}/\mathfrak{q}_i \to 0$$

$$0 \to SPE(\mathfrak{g}) \to E(\mathfrak{g}) \to \bigoplus_{i=1}^{s} E(\mathfrak{g})|_{c_i}/\mathfrak{n}_i \to 0.$$

These are called the sheaves of **parabolic** and **strongly parabolic** sections of the adjoint bundle, respectively.

Definition 1.77 *A* ***parabolic*** G***-Higgs bundle*** *with weights* α *over* (C, D) *is a triple* (E, φ, Q) *where:*

- (E, Q) *is a parabolic* G*-bundle with weights* α *over* (C, D)*.*
- $\varphi \in H^0(PE(\mathfrak{g}) \otimes K_C(D))$ *is a section called* ***Higgs field****.*

If $\varphi \in H^0(SPE(\mathfrak{g}) \otimes K_C(D))$*, we say that the parabolic* G*-Higgs bundle is* ***strongly parabolic****.*

As in the non-parabolic setting, we want to study the moduli space $\mathcal{M}(G, \alpha)$ parametrising isomorphism classes of polystable parabolic G-Higgs bundles. For this, we first recall the suitable notions of stability [15, Section 4.2] [123, Definition 2.2].

Recall from Section 1.2 that given a principal G-bundle E and a reduction $\sigma \in H^0(E(G/P))$ we have a notion of **degree** of σ with respect to the dominant character $\chi \in X_+^*(P)$, given by

$$\deg(E, \sigma, \chi) := \deg \chi(E_\sigma).$$

Moreover, taking into account the parabolic structure Q on E, we may define a **parabolic degree**. For this we add an extra term depending on the parabolic structures and weights. Notice

that in $E|_{c_i}$ we have both a P-orbit $\sigma(c_i)$ and a B-orbit Q_i, therefore there is a well-defined **relative position** by writing $\sigma(c_i) = gP$ and $Q_i = g'B$ in any trivialisation, and taking the element

$$(g, g') \in G\backslash(G \times G)/(P \times B) \simeq P\backslash G/B \simeq W_P\backslash W,$$

where $W_P \subseteq W$ denotes the *parabolic subgroup* corresponding to the Weyl group of L (note that this is indeed a parabolic subgroup in the sense of Definition 1.4). This does not depend on the chosen trivialisation. We denote this relative position by $\deg(\sigma, Q_i)$.

Definition 1.78 *Let (E, Q) be a parabolic G-bundle over (C, D) with weights α, denote by $\sigma \in H^0(E(G/P))$ a reduction of structure group and let $\chi \in X^*_+(P)$ a dominant character. We define its **parabolic degree** as*

$$\operatorname{pardeg}(E, Q, \sigma, \chi) := \deg(E, \sigma, \chi) + \sum_{i=1}^{s} \chi(\deg(\sigma, Q_i)\alpha_i).$$

Notice that the rightmost part of the expression is well defined: since χ is a character of P, the expression does not depend on the chosen representative in $W_P\backslash W$ acting on α, as elements in W_P fix χ.

Remark 1.79 *The previous quantity can also be defined using analytic techniques. On one hand, the degree can be defined via Chern–Weil theory as explained in Remark 1.11. On the other hand, the local part of the parabolic degree can be defined using the Tits distance on the visual boundary of a symmetric space associated to the local data, see [15, Appendix B].*

Now we define the stability notions for parabolic G-Higgs bundles.

Definition 1.80 *A parabolic G-Higgs bundle (E, φ, Q) with weights α over (C, D) is*

- ***semistable**, if for any standard parabolic subgroup $P \subseteq G$, dominant character $\chi \in X^*_+(P)$ and reduction $\sigma \in H^0(C, E(G/P))$ such that $\varphi|_{C\smallsetminus D} \in H^0(C \smallsetminus D, E_\sigma(\mathfrak{p}) \otimes K_C|_{C\smallsetminus D})$, we have*

$$\operatorname{pardeg}(E, Q, \sigma, \chi) \leqslant 0.$$

- ***stable**, if for any proper standard parabolic subgroup $P \subseteq G$, nontrivial dominant character $\chi \in X^*_+(P)\smallsetminus\{0\}$ and reduction $\sigma \in H^0(C, E(G/P))$ such that $\varphi|_{C\smallsetminus D} \in H^0(C\smallsetminus D, E_\sigma(\mathfrak{p})\otimes K_C|_{C\smallsetminus D})$, we have*

$$\operatorname{pardeg}(E, Q, \sigma, \chi) < 0.$$

- ***polystable**, if it is semistable and, for any standard parabolic subgroup $P \subseteq G$, dominant character $\chi \in X^*_+(P)$ and reduction $\sigma \in H^0(C, E(G/P))$ such that $\varphi|_{C\smallsetminus D} \in H^0(C \smallsetminus D, E_\sigma(\mathfrak{p}) \otimes K_C|_{C\smallsetminus D})$ and*

$$\operatorname{pardeg}(E, Q, \sigma, \chi) = 0,$$

there exists a further reduction $\sigma' \in H^0(C, E_\sigma(P/L))$ such that $\varphi \in H^0(C\smallsetminus D, E_{\sigma'}(\mathfrak{l})\otimes K_C|_{C\smallsetminus D})$ and a parabolic structure on the L-bundle $E_{\sigma'}$ inducing that of E.

Using these notions, a **moduli space** of isomorphism classes of polystable parabolic G-Higgs bundles with weight α can be constructed. We denote it by $\mathcal{M}(G, \alpha)$. It is a complex, quasiprojective variety with a Poisson structure, foliated by symplectic leaves corresponding to fixing the extra data of the orbit for the associated graded to the residues at the parabolic punctures [15, Proposition 7.1], [24,94,95]. More precisely, with our assumption that every α_i lies in the interior of the standard Weyl alcove (this can be stated for more general weights as in [15]) we can fix a Levi factor $\mathfrak{t}_i \subseteq \mathfrak{q}_i$,

which is conjugate to $\mathfrak{t}$, and let $\operatorname{Gr}\operatorname{Res}_{c_i}(\varphi) \in \mathfrak{t}_i$ be the projection of the residue of φ at c_i to said Levi factor. Letting $T_i \subseteq Q_i'$ denote the corresponding Levi factor of the Borel subgroup $Q_i' \subseteq E(G)|_{c_i}$, we get a map

$$\mathcal{M}(G,\alpha) \to \prod_{i=1}^{s} \mathfrak{t}_i/T_i.$$

The symplectic leaves of $\mathcal{M}(G,\alpha)$ are given by the fibres of this map. In particular, the locus of strongly parabolic Higgs bundles (which corresponds to the fibre over 0) defines a symplectic leaf

$$\mathcal{M}_{sp}(G,\alpha) \subseteq \mathcal{M}(G,\alpha)$$

which has a hyperkähler structure, and thus it is equipped with a holomorphic symplectic form $\omega \in H^0(\Omega^2_{\mathcal{M}_{sp}(G,\alpha)})$. It has dimension

$$\dim \mathcal{M}_{sp}(G,\alpha) = 2\dim(G)(g-1) + 2s\dim(G/B).$$

Moreover, there ingredients present in the non-abelian Hodge correspondence of Section 1.2 (the Hitchin–Kobayashi correspondence and the Corlette–Donaldson theorem regarding harmonic metrics and local systems) can be generalised to the parabolic setting [15], resulting in a version of the non-abelian Hodge correspondence concerning the character variety for the punctured curve $C \smallsetminus D$. More precisely, we define:

$$\mathcal{R}_p(G) := \operatorname{Hom}(\pi_1(C \smallsetminus D), G) /\!\!/ G$$

to be the character variety parametrising conjugacy classes of completely reducible representations of $\pi_1(C \smallsetminus D)$ into G. Then, fixing conjugacy classes $\mathcal{C} := (\mathcal{C}_1, \dots, \mathcal{C}_s)$, $\mathcal{C}_i \in G/G$ for the image of the generators of $\pi_1(C \smallsetminus D)$ given by the loops around each c_i under a given representation, we get subvarieties

$$\mathcal{R}(G,\mathcal{C}) \subseteq \mathcal{R}_p(G).$$

Theorem 1.81 (Parabolic non-abelian Hodge correspondence [15, Theorem 7.8]). *Given the symplectic leaf $\mathcal{M}_{\mathcal{L}}(G,\alpha) \subseteq \mathcal{M}(G,\alpha)$ corresponding to $\mathcal{L} \in \prod_{i=1}^{s} \mathfrak{t}_i/T_i$, there is a vector $\mathcal{C}_{\mathcal{L}}$ of conjugacy classes (computed in [15, Table 1] and depending injectively on $\mathcal{L}$) and a homeomorphism*

$$\mathcal{M}_{\mathcal{L}}(G,\alpha) \simeq \mathcal{R}(G,\mathcal{C}_{\mathcal{L}}).$$

In particular, when $\mathcal{L} = 0$, there is a homeomorphism

$$\mathcal{M}_{sp}(G,\alpha) \simeq \mathcal{R}(G,(\exp(2\pi\sqrt{-1}\alpha_1),\dots,\exp(2\pi\sqrt{-1}\alpha_s))).$$

The given theorem does not cover all possible conjugacy class vectors $\mathcal{C}$ because we are assuming that the weights α_i belong to the interior of the standard Weyl alcove, but the general statement of [15, Theorem 7.8] does. Note that the strongly parabolic condition corresponds to fixing the conjugacy classes of the representations in the maximal compact subgroup $K \subseteq G$.

We now give a more precise definition of the holomorphic symplectic form ω mentioned above. For this, we need to understand the deformation theory of strongly parabolic G-Higgs bundles.

Definition 1.82 *Let $(E,\varphi,Q) \in \mathcal{M}_{sp}(G,\alpha)$ be a strongly parabolic G-Higgs bundle. Its **deformation complex** is the following complex of sheaves over C:*

$$C^\bullet(E,\varphi,Q) : PE(\mathfrak{g}) \xrightarrow{[\varphi,-]} SPE(\mathfrak{g}) \otimes K_C(D).$$

Notice that indeed $PE(\mathfrak{g})$ maps to $SPE(\mathfrak{g}) \otimes K_C(D)$ because $\varphi \in H^0(SPE(\mathfrak{g}) \otimes K_C(D))$ and $[\mathfrak{q}_i, \mathfrak{n}_i] \subseteq \mathfrak{n}_i$. This complex governs the deformations of (E, φ, Q) in the moduli space.

Proposition 1.83 ([132, Remark 2.1, Proposition 4.2], [95, Section 6]). *The space of deformations of a strongly parabolic G-Higgs bundle (E, φ, Q) is naturally isomorphic to $\mathbb{H}^1(C^\bullet(E, \varphi, Q))$. In particular, if $(E, \varphi, Q) \in \mathcal{M}^s_{sp}(G, \alpha)$ is a smooth point, we have a description of its tangent space*

$$T_{(E,\varphi,Q)}\mathcal{M}(G, \alpha) \simeq \mathbb{H}^1(C^\bullet(E, \varphi, Q)).$$

Remark 1.84 *For a point $(E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)$ to be smooth, it has to be stable. Due to the definition of stability and of the parabolic degree, a generic choice of the weights α implies that every point in the moduli space is stable. However, the resulting space may still have singularities of orbifold type at those points whose automorphism group $\mathrm{Aut}(E, \varphi, Q)$ is strictly larger than the centre $Z(G)$ (see [15, Section 7.1]). Although this does not pose significant problems, we will typically work over the smooth locus $\mathcal{M}^s_{sp}(G, \alpha)$.*

The previous deformation theory can be used to describe the symplectic form ω using the standard Serre duality argument. Indeed, the Killing form induces an isomorphism

$$SPE(\mathfrak{g})^* \simeq PE(\mathfrak{g}) \otimes \mathcal{O}(D)$$

so that

$$\mathbb{H}^1(C^\bullet(E, \varphi, Q))^* \simeq \mathbb{H}^1(C^\bullet(E, \varphi, Q)^* \otimes K_C) \simeq \mathbb{H}^1(C^\bullet(E, \varphi, Q))$$

where the first isomorphism is Serre duality and the second is the Killing form pairing explained above. This identification defines ω.

The moduli space $\mathcal{M}_{sp}(G, \alpha)$ is furthermore endowed with an algebraically completely integrable system, the **strongly parabolic Hitchin system** (originally [76] in the non parabolic setting, see [24, 95]), defined in a similar way to the non-parabolic setting: consider the algebra of invariant polynomial functions $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, \ldots, p_r]$, which is a polynomial ring on r generators. Set $d_i := \deg(p_i)$. Then, given a section $\varphi \in H^0(E(\mathfrak{g}) \otimes K_C(D))$, the evaluation $p_i(\varphi) \in H^0(C, K_C^{d_i}(d_i \cdot D))$ is well defined and, moreover, if φ is strongly parabolic, it lies in $H^0(C, K_C^{d_i}((d_i - 1) \cdot D))$ since $\varphi|_{c_i}$ is nilpotent.

Definition 1.85 *The **strongly parabolic Hitchin map** is defined by*

$$\begin{aligned} h^{sp}_G : \mathcal{M}_{sp}(G, \alpha) &\to \mathcal{A}_{sp}(G) := \bigoplus_{i=1}^{r} H^0(C, K_C^{d_i}((d_i - 1) \cdot D)) \\ (E, \varphi, Q) &\mapsto (p_1(\varphi), \ldots, p_r(\varphi)). \end{aligned}$$

It is a proper map of algebraic varieties and the base has half the dimension of the moduli space.

The strongly parabolic Hitchin system h^{sp}_G shares many properties with the usual one h_G. In particular, many of the mirror symmetry phenomena exhibited in Section 1.3 is also present in $\mathcal{M}_{sp}(G, \alpha)$. The physical intuition for this behaviour comes from the consideration of additional *surface operators* in the starting four dimensional supersymmetric $N = 4$ Yang–Mills theory [66]. Roughly speaking, recall from Section 1.3 that mirror symmetry for $\mathcal{M}(G)$ was explained as a dimensional reduction of S-duality for such a four dimensional theory on $C \times \Sigma$. Now, at each $c_i \in D$, we insert a surface $c_i \times \Sigma$ at which the fields are allowed to have singularities: after performing the dimensional reduction, the target of the sigma model will become a moduli space of parabolic Higgs bundles, and S-duality can be carried over to suggest a mirror symmetry picture. From the point of view of the geometric Langlands correspondence, this models the *ramified* setting

where the G-local systems are allowed to have ramifications and the $G^\vee$-bundles have parabolic structures.

Mathematically, SYZ mirror symmetry for $\mathcal{M}_{sp}(\mathrm{SL}_n(\mathbb{C}),\alpha)$ and $\mathcal{M}_{sp}(\mathrm{PGL}_n(\mathbb{C}),\alpha)$ was proven by Biswas and Dey [18]. While they state it similarly to Theorem 1.31 as a duality of gerbes, in the parabolic setting there exist sections of h_G^{sp} landing in any given connected component, so the gerbes can be trivialised and the symmetry can be simply stated as a duality of abelian fibrations. This was observed in [63, Remark 3.6] and it will be crucial for us, as these sections provide examples of very stable upward flows in $\mathcal{M}_{sp}(G,\alpha)$, see Example 4.28 and Section 4.4 where we compute their mirror branes.

Furthermore, topological mirror symmetry for $\mathcal{M}_{sp}(\mathrm{SL}_n(\mathbb{C}),\alpha)$ and $\mathcal{M}_{sp}(\mathrm{PGL}_n(\mathbb{C}),\alpha)$ was proven in [63, Theorem 3.13] for $n=2,3$ and in [112, Theorem 1.1] for arbitrary n (see also [122]) following a similar p-adic integration strategy as in [64]. The statement is, as in Theorem 1.32, an equality of stringy E-polynomials for both spaces. Note, however, that it is no longer necessary to twist the determinant of the bundles since choosing the weights α generically already produces smooth or orbifold moduli spaces. The statement holds in fact for any such generic choice, even if α does not lie in the interior of the standard alcove.

1.7 Fixed points of the $\mathbb{C}^\times$-action in $\mathcal{M}_{sp}(G,\alpha)$

As in Section 1.4, given a parabolic G-Higgs bundle (E,φ,Q), it can be scaled by a nonzero complex number $\lambda\in\mathbb{C}^\times$:

$$(E,\varphi,Q)\mapsto(E,\lambda\varphi,Q).$$

This scaling results in a well-defined parabolic G-Higgs bundle (since the sheaf of parabolic sections is invariant by scaling) and the stability notions are preserved. Moreover, some of the symplectic leaves are preserved, including $\mathcal{M}_{sp}(G,\alpha)$. This results in a well defined $\mathbb{C}^\times$-action

$$\begin{aligned}\mathcal{M}_{sp}(G,\alpha)&\to\mathcal{M}_{sp}(G,\alpha)\\(E,\varphi,Q)&\mapsto(E,\lambda\varphi,Q).\end{aligned}$$

The Hitchin map h_G^{sp} is $\mathbb{C}^\times$-equivariant for the $\mathbb{C}^\times$-action on $\mathcal{A}_{sp}(G)$ scaling each of the direct summands with weight $d_i=\deg(p_i)$. Also, it is not hard to see from this description that, exactly as in the non-parabolic setting, the $\mathbb{C}^\times$-action maps the holomorphic symplectic form ω defined above to $\lambda\omega$.

As for $\mathcal{M}(G)$, we are also interested in the study of the $\mathbb{C}^\times$-action on $\mathcal{M}_{sp}(G)$: first, we describe the classification of its fixed points. In order to do so, we first set the parabolic structure aside, leading to the following definition.

Definition 1.86 *A $K_C(D)$-**twisted** G-Higgs bundle over C is a pair (E,φ) where E is a principal G-bundle over C and $\varphi\in H^0(E(\mathfrak{g})\otimes K_C(D))$.*

Note that, since for our choice of parabolic weights we have $PE(\mathfrak{g})\subseteq E(\mathfrak{g})$, forgetting the parabolic structure on a parabolic G-Higgs bundle results in a $K_C(D)$-twisted G-Higgs bundle. Being fixed by the $\mathbb{C}^\times$-action as a $K_C(D)$-twisted G-Higgs bundle (i.e. having a family of automorphisms $(E,\varphi)\simeq(E,\lambda\varphi)$) is weaker than being fixed as a parabolic G-Higgs bundle, since the automorphisms need not be compatible with the parabolic structure. Thus, our strategy to identify the fixed points in $\mathcal{M}(G,\alpha)$ is to first do so for $K_C(D)$-twisted Higgs bundles and then study which parabolic structures can we equip in a compatible way with the corresponding $\mathbb{C}^\times$-family of automorphisms.

Proposition 1.87. *Let (E,φ) be a $K_C(D)$-twisted G-Higgs bundle. There is a family of automorphisms*

$$f_\lambda : (E,\varphi) \xrightarrow{\sim} (E,\lambda\varphi)$$

if and only if there is a Vinberg $\mathbb{C}^\times$-pair $(G_0,\mathfrak{g}_k)$ and a reduction $\sigma \in H^0(C,E(G/G_0))$ such that $\varphi \in H^0(E_\sigma(\mathfrak{g}_k)\otimes K_C(D))$.

Proof. The proof is analogous to usual K_C-twisted Higgs bundles, see [14, Proposition 4.15]. For convenience later, we recall why having such a reduction implies the existence of the f_λ. Introduce the cocharacter $\xi_\bullet : \mathbb{G}_m \to G_{\mathrm{ad}}$ defined by exponentiating the grading element ζ. Since ζ is centralised by G_0, this cocharacter lies in the centre of G_0 and defines an automorphism of E_σ, which acts on the Higgs field as $\mathrm{Ad}_{\xi_\lambda}(\varphi) = \lambda^k\varphi$ because φ takes values in $\mathfrak{g}_k$. Since $k \neq 0$, we get that (E,φ) is isomorphic to $(E,\lambda^k\varphi)$ and hence to $(E,\lambda\varphi)$ for any $\lambda \in \mathbb{C}^\times$. □

Now we investigate which parabolic structures Q on a $K_C(D)$-twisted G-Higgs bundle fixed by the $\mathbb{C}^\times$-action are compatible with the family of automorphisms f_λ, resulting in the identification of the $\mathbb{C}^\times$-fixed parabolic G-Higgs bundles.

Proposition 1.88. *Let (E,φ) be a $K(D)$-twisted Higgs bundle fixed by the $\mathbb{C}^\times$-action. Let $f_\lambda : (E,\varphi) \xrightarrow{\sim} (E,\lambda\varphi)$ denote the family of automorphisms given by Proposition 1.87, i.e. obtained by using $\xi_\lambda \in G_{\mathrm{ad}}$ the cocharacter given by exponentiating the grading element ζ. Let $c \in D$ be one of the parabolic punctures.*

There is an identification between the space of parabolic structures $E|_c/B$ and G/B such that the following are equivalent:

- *Every f_λ is a parabolic automorphism with respect to the parabolic structure $gB \in G/B$.*
- *The parabolic structure gB is fixed by the $\mathbb{C}^\times$-action*

$$gB \mapsto \xi_\lambda gB$$

on G/B.

Proof. Note that $\xi_\lambda \in G_{\mathrm{ad}}$ acts on the flag variety because $Z(G) \subseteq B$. As explained in the proof of Proposition 1.87, we can find $e_\lambda \in E|_c$ a trivialisation via which $f'_\lambda(e_\lambda) = \tilde{\xi}_\lambda$ where $\tilde{\xi}_\lambda \in G$ lifts ξ_λ and $f'_\lambda : E|_c \to G$ is the G-equivariant section corresponding to the bundle automorphism $f_\lambda : E \to E$. This element is the desired trivialisation $E|_c \simeq G$ that identifies $E|_c/B$ with G/B: let Q_g denote the parabolic structure corresponding to $gB \in G/B$.

Recall that f_λ and f'_λ are related as follows: $f_\lambda(e) = e f'_\lambda(e)$. Thus

$$f_\lambda(e_g) = e_g f'_\lambda(e_g) = e_g g^{-1}\tilde{\xi}_\lambda g = e_\lambda \tilde{\xi}_\lambda g.$$

Therefore, we have the following diagram:

$$\begin{array}{ccc} E|_c/B & \xrightarrow{\bar{f}_\lambda|_c} & E|_c/B \\ \downarrow{\scriptstyle \bar{e}_\lambda} & & \downarrow{\scriptstyle \bar{e}_\lambda} \\ G/B & \xrightarrow{\tilde{\xi}_\lambda \cdot -} & G/B \end{array} .$$

and a parabolic structure e_g will be fixed by f_λ if and only if it is fixed by the proposed action. □

Remark 1.89 *We can arrive at the same conclusion applying the definition of parabolic automorphism directly: f_λ is a parabolic automorphism with respect to the parabolic structure in the B-orbit of $e_g \in E|_c$ if and only if $f'_\lambda(e_g) \in B$. But we have*

$$f'_\lambda(e_g) = f_\lambda(e_\lambda g) = g^{-1}\tilde{\xi}_\lambda g,$$

therefore the condition is equivalent to

$$g^{-1}\tilde{\xi}_\lambda g \in B.$$

In other words, it is equivalent to $\tilde{\xi}_\lambda gB = gB$.

Remark 1.90 *From the proof above we deduce that the constructed identification has the following property: let $Q_g = (Q_{g_1}, \dots, Q_{g_s})$ be the parabolic structure corresponding to elements $g_i B \in G/B$ under the previous identification at each of the punctures $c_i \in D$. Then*

$$(E, \varphi, Q_g) \simeq (E, \lambda\varphi, Q_{\xi_\lambda g}),$$

via f_λ, where $\xi_\lambda g$ denotes the diagonal action.

Having understood the action on the space of parabolic structures, we can identify the fixed points. For this we also introduce the Weyl group $W_{G_0} \subseteq W_G := W$ given by $N_{G_0}(T)/T$.

Lemma 1.91. *The fixed point locus under the $\mathbb{C}^\times$-action in Proposition 1.88 is a disjoint union of subvarieties*

$$G_0 wB/B \subseteq G/B$$

indexed by $w \in W_{G_0}\backslash W_G$.

Proof. This is a standard result, see e.g. [33, Section 3.1.7] for the case where $\xi_\bullet$ is a regular cocharacter. In general, we can use the Bruhat decomposition

$$G/B = \bigsqcup_{w\in W} BwB/B$$

deduced in *loc.cit.* This decomposition shows that an element in G/B can be written as uwB/B where $u \in U$ is an element in the unipotent radical $U \subseteq B$. This representation is unique if we select the representative $u \in U \cap wU^{opp}w^{-1}$. From this uniqueness it is easy to see that $\xi_\lambda uwB/B = uw/B$ if and only if u is centralised by ξ_λ, that is, if and only if $u \in G_0$, as desired. Of course, the components $G_0 wB/B$ and $G_0 w'B/B$ are the same if and only if $w \equiv w' \mod W_{G_0}$. □

We are now in position of describing the fixed point locus in the moduli space of parabolic G-Higgs bundles. Given a Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_k)$ and a reduction of structure group $\sigma \in H^0(E(G/G_0))$ of a principal G-bundle E with parabolic structure Q, let

$$PE_\sigma(\mathfrak{g}_k) = E_\sigma(\mathfrak{g}_k) \cap PE(\mathfrak{g})$$

denote the sheaf of parabolic sections passing through $E_\sigma(\mathfrak{g}_k) \subseteq E(\mathfrak{g})$. Define $SPE_\sigma(\mathfrak{g}_k)$ similarly.

Theorem 1.92. *A parabolic G-Higgs bundle $(E, \varphi, Q) \in \mathcal{M}(G,\alpha)$ is fixed by the $\mathbb{C}^\times$-action if and only if there is a Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_k)$ such that*

- *There is a reduction of structure group $\sigma \in H^0(E(G/G_0))$.*
- *The Higgs field satisfies $\varphi \in H^0(PE_\sigma(\mathfrak{g}_k) \otimes K_C(D)) \subseteq H^0(PE(\mathfrak{g}) \otimes K_C(D))$.*

- *The parabolic structures satisfy*

$$Q_i \in \bigsqcup_{w \in W_{G_0} \backslash W} \left((E_\sigma)|_{c_i} \cdot w \cdot B \right) / B \subseteq E|_{c_i}/B.$$

Proof. Follows directly from Proposition 1.87, Proposition 1.88 and Lemma 1.91. □

Remark 1.93 *Note that for $k \neq 0$ we have $SPE_\sigma(\mathfrak{g}_k) = PE_\sigma(\mathfrak{g}_k)$. Indeed, elements $X \in \mathfrak{g}_k$ are nilpotent because $\mathrm{ad}^m(X)$ increases the degree by mk, and degrees are bounded since $\mathfrak{g}$ is finite dimensional. Moreover, since our parabolic weights are in the interior of the standard Weyl alcove, nilpotent elements in $\mathfrak{q}_i$ are automatically in $\mathfrak{n}_i$ (this would not be true for other choices of weights). In particular, all fixed points of the $\mathbb{C}^\times$-action lie in $\mathcal{M}_{sp}(G, \alpha)$.*

Example 1.94 *Let us illustrate the above theorem with a simple example. Let $G = \mathrm{PGL}_2(\mathbb{C})$ and choose any weight in the interior of the standard Weyl alcove. The Lie algebra is $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Letting (e, h, f) be a triple of generators satisfying the usual relations, we can grade it by $\zeta = \frac{h}{2}$. Then, we have*

$$G_0 = (\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{C}^\times = T$$

(where the quotient is by the diagonal action) embedded diagonally in $\mathrm{PGL}_2(\mathbb{C})$ and acting on $\mathfrak{g}_1 = \mathbb{C}$ by $(\lambda, \beta) \cdot z = \lambda\beta^{-1} z$. A G_0 bundle is equivalent via the standard representation to a vector bundle

$$E = L_0 \oplus L_1$$

splitting as a direct sum of two line bundles and up to tensoring by any line bundle. Therefore we can normalise it to

$$E = \mathcal{O}_C \oplus L$$

where L is any line bundle (although stability requires $\deg L \leqslant 0$).

A Higgs field taking values in $\mathfrak{g}_1$ is the same as a section $\varphi \in H^0(LK_C(D))$. Finally, a parabolic structure at a point $c \in C$ is equivalent to the choice of a line in the fibre $E|_c$. According to the theorem, the resulting Higgs bundle is fixed if and only if the line belongs to either $(E_\sigma)|_{c_i} B/B$ or $(E_\sigma)|_{c_i} wB/B$ where $w \in W$ is the nontrivial element and B has the Lie algebra spanned by (e, h). The first choice corresponds to the line $L|_c \subseteq E|_c$ and is always compatible with the Higgs field. The second choice corresponds to the line $\mathcal{O}_C|_c \subseteq E|_c$ and requires the section φ to vanish at c.

In any case, it is clear that the resulting Higgs bundle is fixed by the $\mathbb{C}^\times$-action, since E has an automorphism given by scaling L by a factor of λ: this maps φ to $\lambda\varphi$ and preserves any of those two lines in $E|_c$. Note that no other lines are preserved.

Example 1.95 *Let us consider more generally what is the description of fixed points for $G = \mathrm{PGL}_n(\mathbb{C})$. Recall from Example 1.36 how gradings of $\mathfrak{sl}_n(\mathbb{C})$ are described: set V a n-dimensional vector space over $\mathbb{C}$, so that $\mathfrak{g} = \mathfrak{sl}(V)$. Gradings are determined by ordered partitions $(r_k)_k$ of n. Once such a partition is selected, fix a splitting*

$$V = \bigoplus_{k=0}^{m-1} V_k$$

in vector subspaces of dimensions r_k. This determines the grading

$$\mathfrak{g}_p := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}(V_k, V_{k+p})_0$$

where the subscript 0 denotes morphisms that are traceless inside of $\mathfrak{g}$ and is only meaningful when $p = 0$. We have

$$G_0 = \left(\prod_{k=0}^{m-1} \mathrm{GL}(V_k)\right)/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts diagonally by scalars.

A strongly parabolic $\mathrm{PGL}_n(\mathbb{C})$-Higgs bundle (for any choice of weights in the interior of the Weyl alcove, which in this case corresponds to choosing real numbers $\alpha_i^1 > \alpha_i^2 > \dots \alpha_i^n$ such that $\sum_j \alpha_i^j = 0$ and $\alpha_i^1 - \alpha_i^n < 1$ at each $c_i \in D$) is given by

- *A vector bundle E of rank n (up to line bundle tensoring).*
- *A choice of a full flag*
$$Q_i^0 = 0 \subsetneq Q_i^1 \subsetneq \cdots \subsetneq Q_i^n = E|_{c_i}$$
at each $c_i \in D$.
- *A bundle morphism $\varphi : E \to E \otimes K_C(D)$ such that*
$$\varphi(Q_i^j) \subseteq Q_i^{j-1} \otimes K_C(D).$$

For such a strongly parabolic G-Higgs bundle to be a fixed point associated to the previous grading (say, for the pair $(G_0, \mathfrak{g}_1)$, the others being analogous), the vector bundle has to split

$$E = \bigoplus_{k=0}^{m-1} E_k$$

in vector subbundles, the Higgs field has to map

$$\varphi(E_k) \subseteq E_{k+1} \otimes K_C(D)$$

and the parabolic structures need to be graded with respect to the splitting, in other words:

$$Q_i^j = \bigoplus_{k=0}^{m-1} \left(E_k|_{c_i} \cap Q_i^j\right)$$

has to hold for every $i \in \{1,\dots,s\}$ and $j \in \{1,\dots,n\}$.

Remark 1.96 *Theorem 1.92 could be rephrased in a similar fashion to [14, Proposition 4.15], in terms of parabolic $(G_0, \mathfrak{g}_k)$-Higgs pairs as in Section 1.4. This would require defining moduli spaces of parabolic $(G_0, \mathfrak{g}_k)$-Higgs pairs, $\mathcal{M}(G_0, \mathfrak{g}_k, \alpha)$.*

These parametrise triples (E, φ, Q) where E is a principal G_0-bundle, $Q_i \in E|_{c_i}/(G_0 \cap B)$ is a choice of parabolic structure at $c_i \in D$ and $\varphi \in H^0(PE(\mathfrak{g}_k) \otimes K_C(D))$ (here we define $PE(\mathfrak{g})$ by including $E|_{c_i}/(G_0 \cap B)$ in $E(G)|_{c_i}/B$ and then define $PE(\mathfrak{g}_k)$ by intersecting with $E(\mathfrak{g}_k)$).

Then, for any choice of $w_i \in W_{G_0} \backslash W_G$ for each $c_i \in D$, resulting in $w = (w_1, \dots, w_s)$, we have the map

$$\mathcal{M}(G_0, \mathfrak{g}_k, w(\alpha)) \to \mathcal{M}(G, \alpha)$$

by

$$E \mapsto E(G)$$
$$\varphi \mapsto \varphi$$
$$Q_i \mapsto Q_i \cdot w.$$

By the previous Theorem 1.92, the union of the images of all these maps covers the fixed point locus of the $\mathbb{C}^\times$-action. The agreement of both stability conditions is checked as in the discussion preceding [14, Proposition 4.15] by noticing the equivalence of the gauge-theoretic equations involved in both moduli spaces.

Remark 1.97 *For more general parabolic weights, giving parabolic subgroups $P_\alpha \subseteq G$ other than the Borel, it is possible to carry the same analysis to conclude that if the parabolic structure belongs (under the identification in Proposition 1.87) to $G_0 w P_\alpha / P_\alpha$ for $w \in W_{G_0} \backslash W_G / W_{P_\alpha}$, the resulting point is fixed. However, as observed in [15, Section 4], these more general weights require considering the notion of meromorphic equivalence of the principal G-bundles instead of isomorphism, so a priori not every fixed point is obtained in this way. As stated in Remark 1.75, in this situation we would consider more natural to work with parahoric torsors and logahoric Higgs torsors for the corresponding weights α. In that case, the fixed point locus is described by reductions (as a logahoric Higgs torsor!) to Vinberg pairs $(\mathcal{G}_{\alpha,0}, \underline{\mathfrak{g}_k})$ determined by gradings of the scheme of Lie algebras corresponding to the Bruhat–Tits group scheme $\mathcal{G}_\alpha$ for weights α.*

To conclude the discussion about fixed points, we analyse the correct analogue for fixed points of Borel type in this parabolic setting. As in Section 1.4, we want to identify the components of the fixed point locus where the Higgs field is generically a regular (nilpotent) element of the Lie algebra.

Definition 1.98 *A fixed point $(E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$ is of **Borel type** if it verifies the condition of Theorem 1.92 for the Vinberg $\mathbb{C}^\times$-pair $(T, \mathfrak{g}_1)$ corresponding to the Borel grading of Definition 1.37, and φ takes values in $\Omega \subseteq \mathfrak{g}_1$ generically.*

Given a fixed point of Borel type (E, φ, Q) we can associate certain Lie-theoretic data encoding the vanishing multiplicities of φ and the parabolic structures Q. Denote by E_T the reduction of E to T. First, as in the non-parabolic setting, since T acts on each simple root space $\mathfrak{g}_{\beta_j}$, we may extract sections of line bundles by projecting:

$$\varphi_j := \pi_j(\varphi) \in H^0(E_T(\mathfrak{g}_{\beta_j}) \otimes K_C(D)),$$

where we used

$$\pi_j : H^0(E_T(\mathfrak{g}_1) \otimes K_C(D)) \to H^0(E_T(\mathfrak{g}_{\beta_j}) \otimes K_C(D)).$$

These are the **components** of φ along the direction of the simple root β_j. They are non-vanishing sections of the line bundle $E_T(\mathfrak{g}_{\beta_j}) \otimes K_C(D)$: denote by $\mathrm{ord}_c(\varphi_j)$ the order of vanishing.

Definition 1.99 *Given a fixed point $(E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$ of Borel type, we define its **multiplicity coweight at** $c \in C$ by the expression*

$$\mu_c(E, \varphi) := \sum_{j=1}^{r} \mathrm{ord}_c(\varphi_j) \cdot \omega_j^\vee \in P^+_{\Delta^\vee}.$$

It is an element of the space of dominant coweights and it does not depend on Q.

Additionally, Theorem 1.92 indicates that the parabolic structure comes from

$$E_T|_{c_i} \cdot w_i \cdot B/B \subseteq E|_{c_i}/B$$

for some $w_i \in W$. Notice that the space on the left is a single point, so we can naturally associate the Weyl group element $w_i \in W$ to (E, φ, Q) at each $c_i \in D$ given by the parabolic structure. We introduce the following notation:

$$u_c(E, \varphi, Q) := \begin{cases} w_i & c = c_i, \\ 1 & c \notin D \end{cases} \in W.$$

There is a natural way to package all the information above using the extended affine Weyl group.

Definition 1.100 *Given a smooth fixed point $(E,\varphi,Q)\in\mathcal{M}_{sp}(G,\alpha)^{\mathbb{C}^\times}$ of Borel type, we define its **twisted multiplicity divisor** by*

$$w(E,\varphi,Q):=\sum_{c\in C} w_c(E,\varphi,Q)\cdot c$$

where

$$w_c(E,\varphi,Q):=(-\mu_c(E,\varphi))\cdot u_c(E,\varphi,Q)\in\widetilde{W}=P_{\Delta^\vee}\rtimes W.$$

Notice that it is a divisor on C whose coefficients lie in $\widetilde{W}$. Only finitely many points in c have nontrivial coefficient (they are contained in the union of D with the locus where φ is non-regular). If $c\in C\smallsetminus D$, the Weyl group element is trivial and it is more natural to think of the coefficient as an element in $P^-_{\Delta^\vee}\simeq W\backslash\widetilde{W}/W$.

Remark 1.101 *Note that, in the definition of the twisted multiplicity divisor, the coweight part given by $\mu_c(E,\varphi)$ is inverted. This is related to the convention explained in Section 1.1 where $P_{\Delta^\vee}$ acts on $\mathfrak{t}$ by subtracting. This is merely a choice since there is an automorphism of $\widetilde{W}$ given by inverting on the $P_{\Delta^\vee}$ factor as the latter is abelian. The reason for this choice will become clearer in Section 4.1 when $\widetilde{W}$ is regarded inside the loop group $G((z))$.*

Remark 1.102 *Since $\mu_c(E,\varphi)$ is dominant, the part of $w_c(E,\varphi,Q)$ in $P_{\Delta^\vee}$ is always antidominant. Similarly, recalling the definition of $u_{c_i}(E,\varphi,Q)$, the compatibility condition between $\varphi|_{c_i}$ and Q_i can be cleanly restated in terms of combinatorics (recall the notation of Section 1.1) as*

$$u_{c_i}(E,\varphi,Q)\in {}^{J_{\mu_{c_i}(E,\varphi)}}W.$$

*These two conditions determine a subset of $\widetilde{W}$ where $w_{c_i}(E,\varphi,Q)$ must lie. This subset is precisely the right parabolic quotient ${}^{S_W}\widetilde{W}$, of minimal length representatives of $W\backslash\widetilde{W}$, sometimes referred to as **spherical quotient**.*

Remark 1.103 *As in Remark 1.46, for a fixed point (E,φ,Q) of Borel type the topological type $d_E\in\pi_1(T)\simeq X_*(T)$ can also be recovered from $\deg(\mu(E,\varphi)):=\sum_{c\in C}\mu_c(E,\varphi)\in P_{\Delta^\vee}$. Indeed, we have $E(\mathfrak{g}_{\beta_j})\otimes K_C(D)\simeq\mathcal{O}_C(\sum_{c\in C}\beta_j(\mu_c(E,\varphi)))$, which automatically forces*

$$d_E=\deg(\mu(E,\varphi))+\mu_s$$

where $\mu_s=-(2g-2+s)\zeta=-(2g-2+s)\sum_{j=1}^r\omega_j^\vee$.

Remark 1.104 *Moreover, as indicated in Remark 1.47 in the non-parabolic setting, knowing $w(E,\varphi,Q)$ it is possible to recover $(E/Z(G),\varphi,Q)$, i.e recover (E,φ,Q) up to a twist by a $Z(G)$-bundle (therefore uniquely if G is of adjoint type). From the data of $\mu_c(E,\varphi)$ one recovers the line bundles $E_T(\mathfrak{g}_{\beta_j})$ and their sections $\pi_j(\varphi)$. Since the simple roots span the root lattice, knowing $E_T(\mathfrak{g}_{\beta_j})$ recovers $E_T/Z(G)$ and therefore E_T is recovered up to a twist of a $Z(G)$-bundle, and so is $E=E_T(G)$. The Higgs field is completely recovered from the $\pi_j(\varphi)$, and the parabolic structures are completely recovered from the $u_c(E,\varphi,Q)$. On the other hand, it is not always possible to construct (E,φ,Q) with arbitrarily prescribed $w(E,\varphi,Q)$. One immediate constraint is that the degree of the divisor (i.e. the sum of the coefficients) needs to lie in the Iwahori–Weyl group $\widetilde{W}_G$ (this is always satisfied if $G=G_{\mathrm{ad}}$ is of adjoint type). Even then, the resulting bundle may not be stable. Another constraint is provided by Remark 1.102 involving the compatibility with the Higgs field.*

As in the non-parabolic setting, stable fixed points of Borel type are automatically simple, therefore it suffices to consider stability in order to check whether a given fixed point of Borel type determines a smooth point in $\mathcal{M}_{sp}(G,\alpha)$.

Proposition 1.105. *Let (E,φ,Q) be a stable parabolic $\mathbb{C}^\times$-fixed point of Borel type. Then, (E,φ,Q) is simple. In particular, $(E,\varphi,Q)\in\mathcal{M}_{sp}(G,\alpha)^s$.*

Proof. The proof is completely analogous to that of Proposition 1.49. □

Similar to Proposition 1.50, stability (therefore smoothness of the fixed point in the moduli space) can be translated into a condition involving only the twisted multiplicity divisor $w(E,\varphi,Q)$.

Proposition 1.106. *Let (E,φ,Q) be a strongly parabolic G-Higgs bundle satisfying the condition of Theorem 1.92 for the Borel grading and such that φ takes values in the open orbit $\Omega\subseteq\mathfrak{g}_1$ generically. Let $d_E=\deg(\mu(E,\varphi))+\mu_{\mathrm{s}}\in X_*(T)$ denote the topological type from Remark 1.103. The following are equivalent:*

1. *(E,φ,Q) is stable as a parabolic G-Higgs bundle.*
2. *For all nontrivial $\chi\in X_+^*(T)\setminus\{0\}$ we have $\chi(d_E)+\sum_{i=j}^s\chi(u_{c_j}(E,\varphi,Q)(\alpha_j))<0$.*

The previous equivalences also hold replacing stable by semistable and strict inequalities by non-strict inequalities.

Proof. The proof is analogous to that of Proposition 1.50. Let $P\subseteq G$ be a standard parabolic subgroup and $\chi\in X_+^*(P)\setminus\{0\}$ a nontrivial dominant character. Let $\sigma\in H^0(C,E(G/P))$ be a reduction of structure group such that $\varphi|_{C\setminus D}\in H^0(C\setminus D,E_\sigma(\mathfrak{p})\otimes K_C(D))$. The same argument as in Proposition 1.50 shows that such a σ must be unique, and extending structure group via $T\subseteq P$ and $\mathfrak{g}_1\subseteq\mathfrak{b}\subseteq\mathfrak{p}$ from the reduction to the Vinberg $\mathbb{C}^\times$-pair $(T,\mathfrak{g}_1)$ that (E,φ,Q) possesses gives the desired reduction. To conclude, it suffices to notice that

$$\deg(E,\sigma,\chi)=\chi(d_E)$$

and $\deg(\sigma,Q_i)=u_{c_i}(E,\varphi,Q)$ by definition, so that

$$\operatorname{pardeg}(E,\sigma,\chi)=\chi(d_E)+\sum_{i=j}^{s}\chi(u_{c_j}(E,\varphi,Q)(\alpha_j)).$$

□

We also introduce the following terminology.

Definition 1.107 *We say that the twisted multiplicity divisor $w(E,\varphi,Q)$ of a fixed point of Borel type is* ***reduced*** *if its coefficient at every $c\in D$ is minimal under the Bruhat order of $\widetilde{W}$, and its coefficient at every $c\notin D$ is minimal under the Bruhat order of $W\backslash\widetilde{W}/W=P^-_{\Delta^\vee}$.*

Equivalently, it is reduced if for every $c\in D$ we have $l(w_c(E,\varphi,Q))=0$ (see Proposition 1.7) and for every $c\notin D$ we have that $w_c(E,\varphi,Q)$ is minuscule.

Example 1.108 *Consider the fixed points for $G=\mathrm{PGL}_n(\mathbb{C})$ as described in Examples 1.95 and 1.108. The Borel grading corresponds to the decomposition where each piece has degree one. Therefore, fixed points of Borel type are given by sums of n line bundles, which we may write in the form*

$$E=\mathcal{O}_C\oplus K_C^{-1}(-D+D_1)\oplus K_C^{-2}(-2D+D_1+D_2)\oplus\cdots\oplus K_C^{1-n}((1-n)D+D_1+\cdots+D_{n-1})$$

where $D_i\subseteq C$ are effective divisors. The Higgs field takes the form

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varphi_1 & 0 & \dots & 0 & 0 \\ 0 & \varphi_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_{n-1} & 0 \end{pmatrix}$$

where $\varphi_j = \pi_j(\varphi) \in H^0(\mathcal{O}_C(D_j))$ has divisor D_j. The parabolic structure at $c_i \in D$ consists of a full flag obtained by taking at each step one extra term in the decomposition of $E|_{c_i}$ in line bundles given above, subject to the compatibility condition with φ (i.e. not every flag is allowed). The only flag which is always permitted, no matter the Higgs field, corresponds to $u_{c_i}(E,\varphi,Q) = 1$ and is given by

$$\begin{aligned} 0 &\subsetneq K_C^{1-n}((1-n)D + D_1 + \cdots + D_{n-1})|_{c_i} \\ &\subsetneq K_C^{2-n}((2-n)D + D_1 + \cdots + D_{n-2})|_{c_i} \oplus K_C^{1-n}((1-n)D + D_1 + \cdots + D_{n-1})|_{c_i} \\ &\subsetneq \cdots \subsetneq E|_{c_i}. \end{aligned}$$

The twisted multiplicity divisor in this case is obtained by noticing that $u_{c_i}(E,\varphi,Q) \in W \simeq S_n$ corresponds to the permutation of the n factors that brings the previous flag to the parabolic structure over c_i. Then, we have

$$w(E,\varphi,Q) = \left(\sum_{j=1}^{r} -D_j\omega_j^\vee\right) \cdot \sum_{c\in C} u_c(E,\varphi,Q)$$

where the product is taken coefficient-wise.

For a more concrete example, take $n = 2$, consider $D = c_1$ a single point, let $c \neq c_1$ be another point and take

$$E = \mathcal{O}_C \oplus K_C^{-1}(3c + c_1)$$

$$\varphi = \begin{pmatrix} 0 & 0 \\ s_c^3 s_{c_1}^2 & 0 \end{pmatrix}$$

where $s_c \in H^0(\mathcal{O}_C(c))$ is the canonical section with divisor c, and similarly for s_{c_1}. Take the parabolic structure Q_1 to be given by the line $\mathcal{O}_C|_{c_1}$, which is compatible since $\varphi|_{c_1} = 0$.

The twisted multiplicity divisor for this example is

$$w(E,\varphi,Q) = (-2\omega_1^\vee \cdot s_{\beta_1})c_1 + (-3\omega_1^\vee)c$$

where $s_{\beta_1} \in W$ is the non-trivial element.

As in the non-parabolic setting, even if the moduli space is only quasiprojective, the $\mathbb{C}^\times$-action on $\mathcal{M}_{sp}(G,\alpha)$ is semiprojective, that is, the fixed point locus is projective and, for any $(E,\varphi,Q) \in \mathcal{M}_{sp}(G,\alpha)$, the limit $\lim_{\lambda\to 0}(E,\lambda\varphi,Q)$ always exists.

Indeed, the same abstract argument as for $\mathcal{M}(G)$ can be ran: since the Hitchin map h_G^{sp} is $\mathbb{C}^\times$-equivariant and the only fixed point in $\mathcal{A}_{sp}(G)$ is zero, the fixed point locus defines a closed subvariety of $(h_G^{sp})^{-1}(0)$, and the latter is projective because h_G^{sp} is proper. Therefore, the fixed point locus $\mathcal{M}_{sp}^{\mathbb{C}^\times}(G,\alpha)$ is projective. Moreover, since limits when $\lambda \to 0$ exist on $\mathcal{A}_{sp}(G)$ and h_G^{sp} is proper, the valuative criterion of properness implies that limits when $\lambda \to 0$ exist in $\mathcal{M}_{sp}(G,\alpha)$.

Therefore, we get a Białynicki-Birula partition (recall Section 1.5) of the moduli space according to the limit point.

Definition 1.109 *Let $(E,\varphi,Q)\in\mathcal{M}_{sp}^{\mathbb{C}^\times}(G,\alpha)$. The **upward flow** from (E,φ,Q) is the subvariety*

$$W^+_{(E,\varphi,Q)}:=\left\{(E',\varphi',Q')\in\mathcal{M}_{sp}(G,\alpha):\lim_{\lambda\to 0}(E',\lambda\varphi',Q')=(E,\varphi,Q)\right\}\subseteq\mathcal{M}_{sp}(G,\alpha).$$

Since $\mathcal{M}_{sp}(G,\alpha)$ is semiprojective, upward flows define a Białynicki-Birula partition

$$\mathcal{M}_{sp}(G,\alpha)=\bigsqcup_{(E,\varphi,Q)\in\mathcal{M}_{sp}^{\mathbb{C}^\times}(G,\alpha)}W^+_{(E,\varphi,Q)}.$$

Considering limits in the opposite direction, we also still have the following.

Definition 1.110 *Let $(E,\varphi,Q)\in\mathcal{M}_{sp}^{\mathbb{C}^\times}(G,\alpha)$. The **downward flow** from (E,φ,Q) is the subvariety*

$$W^-_{(E,\varphi,Q)}:=\left\{(E',\varphi',Q')\in\mathcal{M}_{sp}(G,\alpha):\lim_{\lambda\to\infty}(E',\lambda\varphi',Q')=(E,\varphi,Q)\right\}\subseteq\mathcal{M}_{sp}(G,\alpha).$$

As in the non-parabolic setting, since the moduli space is not projective, these do not partition the whole moduli space and they determine its **core**:

$$\mathcal{C}:=\bigsqcup_{(E,\varphi,Q)\in\mathcal{M}_{sp}^{\mathbb{C}^\times}(G,\alpha)}W^-_{(E,\varphi,Q)}.$$

As before, $\mathbb{C}^\times$-equivariance and properness of h^{sp}_G allow to deduce that (the underlying reduced schemes for) $\mathcal{C}$ and $h_G^{-1}(0)$ coincide [70, Section 3.3].

In the following we summarise the properties of this decomposition, which are essentially the same as those explained in Section 1.5 for $\mathcal{M}(G)$.

Proposition 1.111 ([11], [70, Propositions 2.1 and 2.10]). *Let $\mathcal{E}:=(E,\varphi,Q)\in\mathcal{M}_{sp}^{s\mathbb{C}^\times}(G,\alpha)$ be a smooth fixed point of the $\mathbb{C}^\times$-action. Consider the weight decomposition for the induced $\mathbb{C}^\times$-action on the tangent space*

$$T_{\mathcal{E}}\mathcal{M}_{sp}(G,\alpha)=\bigoplus_{k\in\mathbb{Z}}T^k_{\mathcal{E}}.$$

Let

$$T^+_{\mathcal{E}}:=\bigoplus_{k>0}T^k_{\mathcal{E}}\subseteq T_{\mathcal{E}}\mathcal{M}_{sp}(G,\alpha)$$

denote the subspace of positive weights and $T^-_{\mathcal{E}}:=\bigoplus_{k<0}T^k_{\mathcal{E}}$ the subspace of negative weights. Then

- *Upward and downward flows $W^+_{\mathcal{E}},W^-_{\mathcal{E}}\subseteq\mathcal{M}_{sp}(G,\alpha)$ are locally closed subvarieties of $\mathcal{M}_{sp}(G,\alpha)$.*
- *There is an isomorphism of $\mathbb{C}^\times$-varieties $W^+_{\mathcal{E}}\simeq T^+_{\mathcal{E}}$. In particular, upward flows are isomorphic to affine spaces.*
- *Similarly, there is an isomorphism of $\mathbb{C}^\times$-varieties $W^-_{\mathcal{E}}\simeq T^-_{\mathcal{E}}$.*
- *With respect to the holomorphic symplectic form ω from Section 1.6, the subvarieties $W^+_{\mathcal{E}}$ are Lagrangian.*

Therefore, upward flows from smooth fixed points provide holomorphic Lagrangian subvarieties with very simple geometry and a $\mathbb{C}^\times$-action, once again providing examples of BAA-brane supports. It is then, as for $\mathcal{M}(G)$, reasonable to ask whether these are closed Lagrangians (since otherwise the process of closing them down by recursively attaching the upward flows of limit points results in a more complicated geometry).

Definition 1.112 ([70, Def. 2.12, Prop. 2.14, Lemma 4.6]) *We say that a smooth fixed point $\mathcal{E} := (E,\varphi,Q) \in \mathcal{M}_{sp}^{s\mathbb{C}^\times}(G,\alpha)$ is **very stable** if any of the following equivalent conditions is satisfied:*

- *We have $W_{\mathcal{E}}^+ \cap h_G^{-1}(0) = \{\mathcal{E}\}$.*
- *The subvariety $W_{\mathcal{E}}^+ \subseteq \mathcal{M}_{sp}(G,\alpha)$ is closed.*
- *The restriction of the Hitchin map $h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \to \mathcal{A}_{sp}(G)$ is a finite, flat surjective morphism.*

As in Proposition 1.71, we can describe upward flows locally (i.e. the deformations of a fixed point that lie in the upward flow) by taking advantage of the previous identification with the space of positive weights of the tangent space. We record here this description for fixed points of Borel type for later use.

Proposition 1.113. *Let (E,φ,Q) be a smooth fixed point of Borel type and (E',φ',Q') be a point on its upward flow. Let $\{(U_j,t_j)\}$ be a trivialisation of E where the $U_j \subseteq C$ are affine opens, the transition functions $f_{ij} : U_i \cap U_j \to T \subseteq G$ take values in T, the local expressions of the Higgs field $\varphi_i : U_i \to \mathfrak{g}_1 \subseteq \mathfrak{g}$ take values in $\mathfrak{g}_1$ and the local expressions of the parabolic structures take values in $W \subseteq G/B$. Then, (E',φ',Q') is isomorphic to the bundle obtained by deforming*

$$f'_{ij} = (1+\varepsilon_{ij})f_{ij}$$

$$\varphi'_i = \varphi_i + \delta_i$$

(and the local expression for the parabolic structures unchanged) for a Čech cocycle

$$\varepsilon_{ij} \in H^0(U_{ij}, PE(\mathfrak{g}_{<0})),$$

$$\delta_i \in H^0(U_i, SPE(\mathfrak{b}^{opp}) \otimes K_C(D)),$$

where $\mathfrak{g}_{<0} := \bigoplus_{k<0} \mathfrak{g}_k$, representing an element in

$$\mathbb{H}^1(PE(\mathfrak{g}) \to SPE(\mathfrak{g}) \otimes K_C(D)).$$

Proof. Recall that the tangent space at (E,φ,Q) is given in Proposition 1.83 by

$$\mathbb{H}^1(PE(\mathfrak{g}) \to SPE(\mathfrak{g}) \otimes K_C(D)),$$

where the arrow is induced by $\mathrm{ad}(\varphi)$. Since $\varphi \in H^0(SPE(\mathfrak{g}_1) \otimes K_C(D))$, this can be decomposed as

$$\bigoplus_{k\in\mathbb{Z}} \mathbb{H}^1(PE(\mathfrak{g}_k) \to SPE(\mathfrak{g}_{k+1}) \otimes K_C(D)).$$

In our fixed trivialisation, an element in the k-th piece of this hypercohomology corresponds to a Čech cocycle $\varepsilon_{ij} \in H^0(U_{ij}, PE(\mathfrak{g}_k))$, $\delta_i \in H^0(U_i, SPE(\mathfrak{g}_{k+1}))$ satisfying the cocycle condition described in detail in [19]. They correspond to the deformation

$$f'_{ij} = (1+z\varepsilon_{ij})f_{ij},$$

$$\varphi'_i = \varphi_i + z\delta_i.$$

(and unchanged local expression for the parabolic structure) where $z \in \mathbb{C}[z]/z^2$ is the deformation parameter.

We may find the induced weight for this piece of this deformation space by applying the $\mathbb{C}^\times$-action, which corresponds to scaling the Higgs field by λ and hence the local data

$$f'_{ij} = (1 + z\varepsilon_{ij})f_{ij},$$

$$\lambda\varphi'_i = \lambda\varphi_i + \lambda z\delta_i.$$

However this no longer has the form of a deformation as above. We can recover that form by applying the local automorphism given by ξ_λ^{-1}, getting:

$$\xi_\lambda^{-1} f'_{ij} \xi_\lambda = (1 + z\lambda^{-k}\varepsilon_{ij})f_{ij}$$

$$\mathrm{Ad}_{\xi_\lambda^{-1}} \lambda\varphi'_i = \varphi_i + z\lambda^{-k}\delta_i$$

and the same local expression for the parabolic structure since ξ_λ^{-1} fixes $W \subseteq G/B$.

Therefore, we see that the weight on this piece is $-k$ and therefore the deformations corresponding to the upward flow are those with $k < 0$, so the ε_{ij} take values in $\bigoplus_{j<0} \mathfrak{g}_j = \mathfrak{g}_{<0}$ and the t_i take values in $\bigoplus_{j\leqslant 0} \mathfrak{g}_j = \mathfrak{b}^{opp}$. □

We conclude this section by describing the very stable upward flows with trivial twisted multiplicity divisor $w(E, \varphi, Q) = 1$. This is the analogue of Example 1.74.

Example 1.114 *We now explain the construction of the fixed point (E^0, φ^0, Q^0) of Borel type that satisfies $w(E^0, \varphi^0, Q^0) = 1$, i.e. that whose Higgs field φ^0 is everywhere regular and Q^0 is always the trivial parabolic structure (i.e. the one preserved by $\mathfrak{g}_1$). This is called the* ***canonical uniformising Higgs bundle****. We do so assuming that G is of adjoint type, since otherwise they cannot be constructed in general; this will be enough for our purposes.*

Let E_T denote the T-bundle obtained from the reduction of E^0. The fact that φ^0 is everywhere regular implies that all the sections $\pi_j(\varphi^0) \in H^0(E_T(\mathfrak{g}_{\beta_j}) \otimes K_C(D))$ are nowhere vanishing. Therefore, the line bundles $E_T(\mathfrak{g}_{\beta_j})$ are all isomorphic to $(K_C(D))^{-1}$. Since G is of adjoint type, the previous condition determines a unique T-bundle E_T.

Moreover, since all the line bundles are equal, this E_T admits a reduction of structure group to $T' \subseteq T$, the one-dimensional subtorus spanned by the principal cocharacter corresponding to the grading element $\zeta = \sum_{j=1}^r \omega_j^\vee$. Consider the regular nilpotent element $e := \sum_{j=1}^r X_{\beta_j} \in \mathfrak{g}_1$ and the associated principal $\mathfrak{sl}_2$-triple $(2\zeta, e, f)$ obtained by the Jacobson–Morozov theorem.

By construction, since $[\zeta, e] = e$, we have $E_{T'}(\langle e\rangle) \simeq (K_C(D))^{-1}$. Therefore, we have a constant section $e \in H^0(E_{T'}(\langle e\rangle) \otimes K_C(D))$ and, extending the structure group of the pair $(E_{T'}, e)$ to G, we get a $K_C(D)$-twisted G-Higgs bundle (E^0, φ^0) which, by construction, reduces to a $K_C(D)$-twisted $(T, \mathfrak{g}_1)$-Higgs pair (in fact, to a $(T', \langle e\rangle)$-Higgs pair).

Finally, we endow it with the unique parabolic structure Q_{c_i} at each $c_i \in D$ such that $u_{c_i}(E^0, \varphi^0, Q^0) = 1$. The result is a parabolic Higgs bundle (i.e. φ^0 is compatible with this parabolic structure) since $\mathfrak{g}_1 \subseteq \mathfrak{b}$. Note that this is the only possible one, since any non-trivial Weyl group element $u \in W$ verifies $u(e) \notin \mathfrak{b}$.

Now we explicitly construct its upward flow as a section of the Hitchin map, as was done originally in [77]. Given a point $a = (a_1, \ldots, a_r) \in \mathcal{A}_{sp}(G) = \bigoplus_{i=1}^r H^0(C, K_C^{\deg(p_i)}((\deg(p_i) - 1)D))$ in the Hitchin base, decompose $\mathfrak{g}$ into irreducible representations of the $\mathfrak{sl}_2(\mathbb{C})$-representation induced by $(2\zeta, e, f)$, getting

$$\mathfrak{g} = \bigoplus_{i=1}^r V_i.$$

There are $r = \operatorname{rank}\mathfrak{g}$ summands, and they can be ordered so that $\dim V_i = 2\deg p_i - 1$ where the $p_i \in \mathbb{C}[\mathfrak{g}]^G$ are the chosen basis elements of invariant polynomials for the construction of h_G^{sp}. Thus, if $f_i \in V_i$ is a lowest weight vector, we have that $[2\zeta, f_i] = (-2\deg p_i + 2)f_i$, and hence $E_{T'}(\langle f_i\rangle) \otimes K_C(D) \simeq (K_C(D))^{\deg p_i - 1} \otimes K_C(D) \simeq (K_C(D))^{\deg p_i}$. This means that we have a

well-defined constant section $f_i \in H^0(C, E_{T'}(\langle f_i\rangle)\otimes K_C(D)\otimes (K_C(D))^{-\deg p_i})$ *and hence* $a_i f_i \in H^0(C,E^0(\mathfrak{g})\otimes K_C)\subseteq H^0(E^0(\mathfrak{g})\otimes K_C(D))$. *This results in a well defined Higgs field*

$$\varphi^a := \varphi^0 + \sum_{i=1}^{r} a_i f_i$$

such that $\varphi^a(c_i)=\varphi^0\in\mathfrak{g}_1$. *In particular, the parabolic structure* Q^0 *is still compatible with all the Higgs fields* φ^a.

The f_i *can be selected so that* $h^{sp}_G(E^0,\varphi^a,Q^0)=a$. *We also have that* (E^0,φ^a,Q^0) *are smooth points ([77, Section 5], or directly from Propositions 1.105 and 1.106, where for the latter we use that* $\chi(\alpha_j)<1$ *and* $2g-2+s\geqslant 1$*) in* $\mathcal{M}_{sp}(G,\alpha)$. *By the* $\mathbb{C}^\times$*-equivariance of* h^{sp}_G, *we get the subvariety*

$$\mathcal{A}_{sp}(G)\simeq\{(E^0,\varphi^a,Q^0): a\in\mathcal{A}_{sp}(G)\}\subseteq W^+_{(E^0,\varphi^0,Q^0)}.$$

However, both sides are vector spaces of the same dimension, hence equality holds. It then follows from this explicit description of the upward flow that (E^0,φ^0,Q^0) *is very stable.*

Notice from Remark 1.104 that when G *is not of adjoint type there may not exist an everywhere regular fixed point of Borel type since the association of the line bundle* $(K_C(D))^{-1}$ *to every simple root need not extend to the cocharacter lattice of* T *(essentially because* $\deg K_C(D)$ *may be odd).*

Chapter 2

Very stable regular G-Higgs bundles

2.1 The affine Grassmannian and Hecke transformations of G-Higgs bundles

The main technique in the classification of very stable Higgs bundles of Borel type is, as in the $G = \mathrm{GL}_n(\mathbb{C})$ case [70, Section 4.2], that of *Hecke transformations*. These modify a Higgs bundle locally around a point to obtain a family of different Higgs bundles in the moduli space which are isomorphic to the starting one away from that point.

A convenient setting to study Hecke transformations, which allows to work with different groups G as well as to perform more types of transformations (even in the $\mathrm{GL}_n(\mathbb{C})$ case), is that of the *affine Grassmannian*. In this section we start by recalling its definition and main properties that we will need for later use. Our main reference is [134].

We will define the affine Grassmannian via its functor of points, which is the natural point of view for Hecke transformations. The functor corresponds to the moduli problem of bundles on a formal disk which are trivial away from its centre, in other words, local modifications of the trivial bundle. Thus, let $\mathbb{D} := \operatorname{Spec} \mathbb{C}[[z]]$ be the formal disk and $\mathbb{D}^* := \operatorname{Spec} \mathbb{C}((z))$ be the punctured disk. We denote by $0 \in \mathbb{D}$ the closed point. For a $\mathbb{C}$-algebra R we also define $\mathbb{D}_R := \operatorname{Spec}(R[[z]])$ and $\mathbb{D}^*_R := \operatorname{Spec}(R((z)))$. We also have a closed subscheme $0_R := \operatorname{Spec} R \hookrightarrow \mathbb{D}_R$.

Definition 2.1 *The **affine Grassmannian** Gr_G is defined by its functor of points: to a $\mathbb{C}$-algebra R it assigns the space of isomorphism classes of pairs (E, Ψ) where E is a principal G-bundle over $\mathbb{D}_R$ and $\Psi : E|_{\mathbb{D}^*_R} \to \mathbb{D}^*_R \times G$ is a trivialisation over $\mathbb{D}^*_R$.*

The above functor of points defines an ind-projective ind-scheme [134, Theorem 1.2.2]. Since we are only interested in the field of complex numbers, we may also regard it as an infinite-dimensional complex-analytic space.

For our purposes it suffices to understand the $\mathbb{C}$-points, which can be done in terms of loop groups. Consider the **loop group** $G((z)) := G(\mathbb{D}^*)$ and the **positive loop group** $G[[z]] := G(\mathbb{D})$. The latter is a subgroup of the former.

Then, the $\mathbb{C}$-points of Gr_G can be identified (see [134, Proposition 1.3.6]) with the quotient $G((z))/G[[z]]$. For a pair (E, Ψ) we can fix a trivialisation $\varepsilon : \mathbb{D} \times G \xrightarrow{\sim} E$, which always exists since the base is $\mathbb{D}$, and then we get an element $\Psi \circ \varepsilon \in \mathrm{Aut}(\mathbb{D}^* \times G) \simeq G((z))$. A change in trivialisation ε' results in a change by $(\varepsilon')^{-1} \circ \varepsilon \in \mathrm{Aut}(\mathbb{D} \times G) \simeq G[[z]]$.

Remark 2.2 *Since we only need the $\mathbb{C}$-points, we will use the previous identifications liberally, in particular denoting by Gr_G the space of $\mathbb{C}$-points. Nevertheless, the previous identifications can be defined schematically as well, by considering the loop groups and their subgroups as schemes (for example, $LG(R) := G(\mathbb{D}_R)$).*

Similarly, when the group is clear from context, we sometimes make use of the notation $\mathrm{Gr} := \mathrm{Gr}_G$.

Now we recall one of the main structural results about Gr_G, the *Cartan decomposition*. We state it for $\mathbb{C}$-points, but the result holds schematically. Given $\mu \in X_*(T)$ a cocharacter, we may regard it as an element $z^\mu \in G((z))$ by taking the map on $\mathbb{C}((z))$-points, $\mu : \mathbb{C}((z))^\times \to T((z)) \subseteq G((z))$, and then taking z^μ to be the image of z.

Definition 2.3 *Let* $\mu \in X_*(T)$. *We define its* ***(affine) Bruhat cell*** *in* Gr_G *as*

$$\mathrm{Gr}_{G,\mu} := G[[z]] \cdot z^\mu \subseteq \mathrm{Gr}_G\,.$$

We define the ***(affine) Schubert variety*** *of* z^μ *in* Gr_G *as the closure*

$$\mathrm{Gr}_{G,\leqslant\mu} := \overline{\mathrm{Gr}_{G,\mu}} \subseteq \mathrm{Gr}_G\,.$$

As we will see in more detail in Chapter 4 in the setting of more general affine flag varieties, the Cartan decomposition is always governed by the Bruhat order of certain parabolic quotient of $\widetilde{W}_G$, as introduced in Section 1.1. For the affine Grassmannian, this parabolic quotient is the double quotient ${}_{S_W}\left(\widetilde{W}_G\right)^{S_W}$, where we recall that $S_W = \{s_1, \ldots, s_r\}$ is the generating set for the Weyl group W. Therefore, the set ${}_{S_W}\left(\widetilde{W}_G\right)^{S_W}$ consists of representatives of $W\backslash\widetilde{W}_G/W = W\backslash(X_*(T) \rtimes W)/W \simeq X_*^+(T)$. It turns out [110, Corollary 1.8] that the representative set ${}_{S_W}(\widetilde{W}_G)^{S_W}$ is precisely $X_*^+(T) \subseteq \widetilde{W}_G$, and its Bruhat order agrees with the usual dominance order, where $\lambda \leqslant \mu$ if and only if $\mu - \lambda$ is a sum of positive coroots. Minimal elements under this order are precisely the minuscule cocharacters.

Proposition 2.4 (Cartan–Iwahori–Matsumoto decomposition [81]). *The affine Grassmannian satisfies the Cartan–Iwahori–Matsumoto decomposition in Bruhat cells:*

$$\mathrm{Gr}_G := \bigsqcup_{\mu \in X_*^+(T)} \mathrm{Gr}_{G,\mu}$$

Furthermore, the Schubert varieties for $\mu \in X_*^+(T)$ *can be described as follows:*

$$\mathrm{Gr}_{G,\leqslant\mu} = \bigsqcup_{\lambda \leqslant \mu} \mathrm{Gr}_{G,\lambda},$$

where we take $\lambda \in X_*^+(T)$ *as well, and their dimension equals the length* $l(z^\mu) = (\sum_{\beta\in\Delta^+}\beta)(\mu)$ *in* $\widetilde{W}_G$.

Affine Bruhat cells are smooth quasi-projective varieties whose closure is the corresponding affine Schubert variety, hence the latter are projective. The fact that Bruhat cells are not closed (unless they correspond to a minuscule cocharacter) translates directly to the classification of very stable G-Higgs bundles: we exploit curves in Bruhat cells whose limit lies outside of the cell in order to construct, via Hecke transformations, nilpotent Higgs bundles in the upward flow of a fixed point of Borel type whose limit lies outside of the upward flow. In order to do that, we need to have an explicit description of curves of this kind. Said description is provided by [134, Proposition 2.1.5].

Lemma 2.5. *Let* $\mu \in X_*(T)$ *be a cocharacter and* $\beta^\vee \in \Delta_+^\vee$ *a positive coroot with corresponding positive root* $\beta \in \Delta^+$. *Assume that* $\beta(\mu) - 1 \geqslant 0$. *Then, there exists a curve* $C_{\mu,\beta} \subseteq \mathrm{Gr}_G$, *isomorphic to* $\mathbb{P}^1_{\mathbb{C}}$, *such that* $C_{\mu,\beta} \cap \mathrm{Gr}_{\mu-\beta^\vee} = \{z^{\mu-\beta^\vee}\}$ *and the remaining* $C_{\mu,\beta} \smallsetminus \{z^{\mu-\beta^\vee}\} \simeq \mathbb{A}^1_{\mathbb{C}}$ *is contained in* Gr_μ.

Proof. Let $m := \beta(\mu) - 1 \geqslant 0$ and $i_\beta : \mathrm{SL}_2 \to G$ the corresponding map from the SL_2-subgroup given by β. The latter naturally induces $Li_\beta : \mathrm{SL}_2((z)) \to G((z))$. Consider the following subgroup isomorphic to SL_2:

$$K_m := \left\{ \begin{pmatrix} a & bz^m \\ cz^{-m} & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\} \subseteq L\,\mathrm{SL}_2\,.$$

The desired curve is $Li_\beta(K_m)z^\mu$. The claims can be checked by considering the subgroup $K_m \cap \mathrm{SL}_2[[z]]$ given by $c = 0$. By multiplication with a diagonal element in SL_2 (which commutes with μ and vanishes in the quotient Gr_G) we get the affine part $Li_\beta \begin{pmatrix} 1 & bz^m \\ 0 & 1 \end{pmatrix} z^\mu \simeq \mathbb{A}^1$, contained in $Li_\beta(\mathrm{SL}_2[[z]])z^\mu \subseteq G[[z]]z^\mu$. On the other hand, the remaining point can be computed [134, Proposition 2.1.5] by working inside $Li_\beta(\mathrm{SL}_2((z)))$, by noticing that

$$Li_\beta \begin{pmatrix} 1 & bz^m \\ 0 & 1 \end{pmatrix} z^\mu = z^\mu Li_\beta \begin{pmatrix} 1 & bz^{m-\beta(\mu)} \\ 0 & 1 \end{pmatrix} = z^\mu Li_\beta \begin{pmatrix} 1 & bz^{-1} \\ 0 & 1 \end{pmatrix}.$$

Then, we can exploit the following identity in $\mathrm{SL}_2((z))$:

$$\begin{pmatrix} 1 & bz^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ b^{-1} & z \end{pmatrix} \begin{pmatrix} z & b \\ -b^{-1} & 0 \end{pmatrix}.$$

The rightmost element lies in $\mathrm{SL}_2[[z]]$ and Li_β maps it to $G[[z]]$. Therefore, it can be disregarded, and the remaining limit point is

$$z^\mu Li_\beta \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = z^\mu z^{-\beta^\vee} = z^{\mu - \beta^\vee},$$

as claimed. □

Since we are interested in G-Higgs bundles, we now bring Higgs fields into the picture of the affine Grassmannian. Note that, when considering Higgs fields over $\mathbb{D}$, they are simply sections in $H^0(\mathbb{D}, E(\mathfrak{g}))$ without further twisting. Denote by $\mathfrak{g}((z)) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ the **loop Lie algebra** and by $\mathfrak{g}[[z]] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[z]]$ the **positive loop Lie algebra**.

Definition 2.6 *Let $X \in \mathfrak{g}((z))$. The moduli space of isomorphism classes of pairs $((E, \varphi), \Psi)$ where (E, φ) is a G-Higgs bundle over $\mathbb{D}$ and $\Psi : (E, \varphi)|_{\mathbb{D}^*} \xrightarrow{\sim} (\mathbb{D}^* \times G, X)$ is a trivialisation over $\mathbb{D}^*$ is called the **affine Springer fibre** of X in Gr_G and denoted Gr_G^X.*

The reason for the name and the notation is that there is an inclusion inside of Gr_G given by the forgetful map that omits φ. Naturally, using the descriptions via quotients of $G((z))$ explained above, we have the following concrete identification:

$$\mathrm{Gr}_G^X := \{\sigma \in \mathrm{Gr}_G : \mathrm{Ad}_{\sigma^{-1}}(X) \in \mathfrak{g}[[z]]\} \subseteq \mathrm{Gr}_G\,.$$

This description follows by taking a given (E, Ψ) and imposing that $\Psi^*(X)$ extends to $0 \in \mathbb{D}$.

Now, using the affine Grassmannian and its affine Springer fibres, we describe Hecke transformations for G-Higgs bundles. First, we understand Hecke transformations of principal G-bundles

(see e.g. [130]) and later on we incorporate the Higgs field. Fix a point $c \in C$ and let $C_0 := C \smallsetminus \{c\}$ be the complement of c and $C_1 := \operatorname{Spec} \mathbb{C}[[z]]$ a formal disk. We have a natural open embedding $C_0 \hookrightarrow C$, as well as a map $C_1 \to C$ defined from the algebraic map

$$\mathcal{O}_C \to \mathcal{O}_{C,c} \to \widehat{\mathcal{O}_{C,c}} \simeq \mathbb{C}[[z]]$$

given by localisation at c and completion, where z is a uniformising parameter at $c \in C$. Let $C_{01} = C_0 \cap C_1 \simeq \operatorname{Spec} \mathbb{C}((z))$ be the intersection.

Now, given a principal G-bundle E, it defines restrictions $E_0 := E|_{C_0}$, $E_1 := E|_{C_1}$ as well as a gluing map

$$\Phi : E_1|_{C_{01}} \xrightarrow{\sim} E_0|_{C_{01}}.$$

By the Beauville–Laszlo theorem [9], the data (E_0, E_1, Φ) is equivalent to E. More precisely, the category of principal G-bundles over C is equivalent to the category of triples (E_0, E_1, Φ) as above, where a morphism between (E_0, E_1, Φ) and (E_0', E_1', Φ') consists of a pair of G-bundle morphisms $(f_0 : E_0 \to E_0', f_1 : E_1 \to E_1')$ making the following diagram commute:

$$\begin{array}{ccc} E_0|_{C_{01}} & \xrightarrow{f_0|_{C_{01}}} & E_0'|_{C_{01}} \\ \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle \Phi'} \\ E_1|_{C_{01}} & \xrightarrow{f_1|_{C_{01}}} & E_1'|_{C_{01}} \end{array}$$

Definition 2.7 *Let E be a principal G-bundle over C. A **Hecke transformation** of E at $c \in C$ is another principal G-bundle E' over C together with an isomorphism*

$$\Psi : E'|_{C_0} \to E|_{C_0}.$$

*Two Hecke transformations (E', Ψ) and (E'', Ψ') are **equivalent** if and only if there is an isomorphism $E' \xrightarrow{f} E''$ such that $\Psi' \circ f|_{C_0} = \Psi$.*

Now we describe the space of Hecke transformations for a fixed principal G-bundle E. For this, fix an isomorphism

$$t_1 : E|_{C_1} \to C_1 \times G$$

with the trivial G-bundle over C_1. This can be done because C_1 is a formal disk. Then, E defines local data (E_0, E_1, Φ') which is isomorphic via (Id, t_1) to the local data $(E_0, C_1 \times G, \Phi)$ with which we work (here $\Phi = \Phi' \circ t_1|_{C_{01}}^{-1}$). Given $\sigma \in \operatorname{Aut}(C_{01} \times G) \simeq G((z))$, we can transform

$$(E_0, C_1 \times G, \Phi) \mapsto (E_0, C_1 \times G, \Phi \circ \sigma).$$

The result is clearly a Hecke transformation, where the isomorphism Ψ is the identity in terms of the local data.

Suppose now that $\sigma, \sigma' \in G((z))$ produce equivalent Hecke transformations, that is, $(E_0, C_1 \times G, \Phi \circ \sigma) \simeq (E_0, C_1 \times G, \Phi \circ \sigma')$. Let (f_0, f_1) denote the corresponding isomorphisms. Since Ψ is the identity automorphism of E_0, we must have $f_0 = \mathrm{Id}$. On the other hand, $f_1 \in \operatorname{Aut}(C_1 \times G) \simeq G[[z]]$. The compatibility condition is

$$\Phi \circ \sigma = \Phi \circ \sigma' \circ f_1$$

so that $\sigma = \sigma' f_1$. In other words, σ and σ' differ by a positive loop $f_1 \in G[[z]]$.

We have seen that given an element of $\mathrm{Gr}_G = G((z))/G[[z]]$ we obtain a Hecke transformation of E, and different elements give inequivalent transformations. We conclude by showing that any

transformation is equivalent to one that can be obtained in this way. Suppose that (E', Ψ) is a Hecke transformation of E. Consider the local data (E'_0, E'_1, Φ'). Take a trivialisation

$$t'_1 := (\Phi') : E'_1 \to C_1 \times G$$

and apply the isomorphism given by (Ψ, t'_1), resulting in isomorphic local data $(E_0, C_1 \times G, \Phi'')$ where $\Phi'' = \Psi|_{C_{01}} \circ \Phi' \circ (t'_1|_{C_{01}})^{-1}$. Now, both Φ'' and Φ are isomorphisms from $C_{01} \times G$ to $E_0|_{C_{01}}$, therefore they must differ by an element of $\mathrm{Aut}(C_{01} \times G) \simeq G((z))$, as desired. A change in the choice of t'_1 results in the same point in the affine Grassmannian since the resulting loops differ by an element in $G[[z]]$. Therefore, we get the following.

Proposition 2.8. *Fixing a trivialisation*

$$t_1 : E|_{C_1} \to C_1 \times G$$

identifies the space of Hecke transformations of E at $c \in C$ up to equivalence with Gr_G.

Before continuing, let us remark that, had we chosen a different trivialisation $t'_1 = \eta \circ t_1$ with $\eta \in \mathrm{Aut}(C_1 \times G) \simeq G[[z]]$, then the local data via t_1, which we denoted by $(E_0, C_1 \times G, \Phi)$, is isomorphic to the local data via t'_1, which is $(E_0, C_1 \times G, \Phi \circ \eta^{-1})$. Therefore the spaces of Hecke transformations are related by a left multiplication by $\eta \in G[[z]]$. This means that if we do not fix any trivialisation, we still have a well defined *type* of the Hecke transformation, as an element in

$$G[[z]] \backslash G((z)) / G[[z]] \simeq X_*^+(T).$$

the space of dominant cocharacters.

Example 2.9 *Consider $G = \mathrm{PGL}_2(\mathbb{C})$ and let $E := \mathcal{O}_C \oplus K_C^{-1}$. For the local trivialisation around C_1, fix one which splits as a direct sum of local trivialisations for each of the two line bundles. Let $\sigma = z^{\omega_1^\vee}$ be the first fundamental cocharacter, namely*

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Twisting the glueing function by the constant map $z \mapsto 1$ does not change the line bundle, and twisting by the identity $z \mapsto z$ corresponds to a twist by $\mathcal{O}_C(-c)$. Therefore, the resulting Hecke transformation is isomorphic to $E' = \mathcal{O}_C(-c) \oplus K_C^{-1}$, where Ψ is the sum of the identity on the K_C^{-1} factor and the isomorphism $\mathcal{O}_C(-c)|_{C \smallsetminus c} \simeq \mathcal{O}_{C \smallsetminus c}$.

Remark 2.10 *It is worth noting that, as it is clear from the previous example, Hecke transformations do not preserve the topological invariant $\varepsilon(E) \in \pi_1(G)$ of the bundle. Take the following convention for the topological invariant: identifying $\pi_1(G) \simeq X_*(T)/Q_{\Delta^\vee}$, given a character $\chi : G \to \mathbb{C}^\times$ we demand $\chi(\varepsilon(E)) = \deg \chi(E)$ where $\chi(\varepsilon(E)) \in \mathrm{Aut}(\mathbb{C}^\times) \simeq \mathbb{Z}$ in the natural way (the integer k corresponds to $z \mapsto z^k$). Then, we have*

$$\varepsilon(E') = \varepsilon(E) - [\sigma]$$

where $[\sigma] \in X_*^+(T)$ is the type of the transformation seen as an element in $\pi_1(G)$. This was remarked in [130, Proposition 1.12] (note that the reference follows the opposite sign convention).

Remark 2.11 *When $G = \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C}), \mathrm{PGL}_n(\mathbb{C})$, there is a well known notion of Hecke transformation (which is used, for example, in the classification of very stable $\mathrm{GL}_n(\mathbb{C})$-Higgs bundles of*

type $(1,1,\dots,1)$ *[70, Section 4.2]) involving choosing a vector subspace of the fibre* $E|_c$ *and performing some sheaf operations. This can be explained as follows in our setting. Whenever* $\omega_j^\vee \in X_*^+(T)$ *is a minuscule cocharacter, the map*

$$\mathrm{Gr}_{z^{\omega_j^\vee}} = G[[z]]z^{\omega_j^\vee} = G[[z]]/(G[[z]] \cap z^{-\omega_j^\vee}(G[[z]])z^{\omega_j^\vee}) \to G/P_j,$$

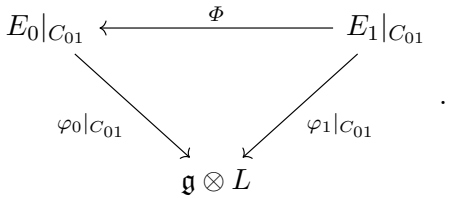
given by evaluation at $0 \in \mathbb{D}$*, where* $P_j := P_{\omega_j^\vee} \subseteq G$ *is the maximal parabolic associated to* $\omega_j^\vee$ *(as explained in Section 1.1), is an isomorphism (see [80, Example 4.12] or [134, Lemma 2.1.13]).*

Therefore, Hecke transformations of minuscule type are given by elements in the flag varieties G/P_j*. In particular, in the case of Dynkin type A, every* $\omega_j^\vee$ *is minuscule,* G/P_j *is isomorphic to a Grassmannian in* $E|_c$ *and hence we get Hecke transformations for each subspace of* $E|_c$ *of any dimension. This notion agrees with the standard one.*

Now we consider the presence of a Higgs field $\varphi \in H^0(E(\mathfrak{g}) \otimes L)$ where L is any line bundle over C (the specific line bundle will not play a role since it is trivial over $\mathbb{D}$, allowing us to use this analysis for parabolic Higgs bundles later in Section 4.1). First we understand the local data. The Higgs field gives

$$\varphi_0 := \varphi|_{C_0} : E_0 \to \mathfrak{g} \otimes L$$
$$\varphi_1 := \varphi|_{C_1} : E_1 \to \mathfrak{g} \otimes L$$

where both maps are G-equivariant. The data must satisfy the compatibility condition



A morphism between local data $(E_0, E_1, \Phi, \varphi_0, \varphi_1)$ and $(E_0', E_1', \Phi', \varphi_0', \varphi_1')$ is again a pair (f_0, f_1) which, on top of being a morphism between (E_0, E_1, Φ) and (E_0', E_1', Φ'), has to satisfy the compatibility conditions

$$\varphi_i' \circ f_i = \varphi_i$$

for $i \in \{0, 1\}$.

Definition 2.12 *Let* (E, φ) *be a G-Higgs bundle over* C*. A* ***Hecke transformation*** *of* (E, φ) *at* $c \in C$ *is another principal G-bundle* E' *over* C *together with an isomorphism*

$$\Psi : E'|_{C_0} \to E|_{C_0}$$

such that $\varphi_0 \circ \Psi = \varphi_0'$*. Two Hecke transformations* (E', φ', Ψ) *and* (E'', φ'', Ψ') *are* ***equivalent*** *if and only if there is an isomorphism* $E' \xrightarrow{f} E''$ *such that* $\Psi' \circ f|_{C_0} = \Psi$ *and* $\varphi'' \circ f = \varphi'$*.*

Note that Hecke transformations for (E, φ) are in particular Hecke transformations for E. Therefore, we may describe them (as before) by fixing a trivialisation

$$t_1 : E_1 \to C_1 \times G.$$

Using t_1 we may assume that our local data is $(E_0, C_1 \times G, \Phi, \varphi_0, \varphi_1)$. Since $\varphi_1 : C_1 \times G \to \mathfrak{g} \otimes L$ is G-equivariant, fixing once and for all a trivialisation of L we may also regard it as an element $\varphi_1 : C_1 \to \mathfrak{g}$ of $\mathrm{Map}(C_1, \mathfrak{g}) \simeq \mathfrak{g}[[z]]$ by evaluating at the identity element of G. This is an identification that we will apply liberally.

As before, let (E', φ', Ψ) be a Hecke transformation with local data $(E'_0, E'_1, \Phi', \varphi'_0, \varphi'_1)$. Taking any trivialisation $t'_1 : E_1 \simeq C_1 \times G$ and applying the isomorphism (Ψ, t'_1) allows us to assume that the local data of the transformation is of the form

$$(E_0, C_1 \times G, \Phi'', \varphi_0, \varphi''_1).$$

(We know that the local expression of the Higgs field over C_0 has to be φ_0 because the Hecke transformation isomorphism over C_0 with respect to this local data is the identity).

Therefore, $\Phi'' = \Phi \circ \sigma$ for some $\sigma \in \mathrm{Aut}(C_{01} \times G) \simeq G((z))$. Then, over C_{01} we are forced to have

$$\varphi''_1 = \varphi_0 \circ \Phi \circ \sigma = \varphi_1 \circ \sigma = \mathrm{Ad}_{\sigma^{-1}} \varphi_1$$

where the last step is due to the G-equivariance of φ_1. Therefore, we must require the condition

$$\mathrm{Ad}_{\sigma^{-1}} \varphi_1 \text{ extends to } C_1.$$

Since $\varphi_1 : C_1 \times G \to \mathfrak{g} \otimes L$, fixing once and for all a trivialisation of L and using G-equivariance we may also regard it as an element of $\mathrm{Map}(C_1, \mathfrak{g}) \simeq \mathfrak{g}[[z]]$. Therefore, we get a Hecke modification for every $\sigma \in G((z))$ such that

$$\mathrm{Ad}_{\sigma^{-1}}(\varphi_1) \in \mathfrak{g}[[z]].$$

As before, two such modifications are equivalent if and only if they differ by a positive loop on the right.

Proposition 2.13. *Fixing a trivialisation*

$$t_1 : E|_{C_1} \to C_1 \times G$$

as well as a trivialisation of L identifies the space of Hecke transformations of (E, φ) at $c \in C$ up to equivalence with the affine Springer fibre $\mathrm{Gr}_G^{\varphi_1}$

Similarly to the previous analysis, changing the trivialisation to $t'_1 = \eta \circ t_1$ results in a change of Springer fibre, since $\mathrm{Ad}_{\eta^{-1}}(\varphi'_1) = \varphi_1$. The previous relation between affine Grassmannians for each trivialisation (multiplying by η) naturally maps one space to the other.

$$\mathrm{Gr}_G^{\mathrm{Ad}_\eta(\varphi_1)} = \eta \, \mathrm{Gr}_G^{\varphi_1} .$$

For our purposes it is especially relevant to consider the case where φ_1 is a nilpotent Higgs field, since we mostly perform Hecke transformations of fixed points of the $\mathbb{C}^\times$-action. In this setting, the following result will be one of the key pieces in the proof of the main theorem.

Lemma 2.14. *Let $\varphi_1 \in \mathfrak{g}[[z]]$ be such that $\varphi_1(0) = e \in \mathfrak{g}$ is a regular nilpotent element of $\mathfrak{g}$. Let $\omega_j^\vee \in P_{\Delta^\vee}$ be a minuscule coweight and assume that $\omega_j^\vee \in X_*(T)$. Then, the intersection*

$$\mathrm{Gr}_G^{\varphi_1} \cap \mathrm{Gr}_{G, z^{\omega_j^\vee}}$$

of the affine Springer fibre of φ_1 with the Bruhat cell of $z^{\omega_j^\vee}$ consists of a single point.

Proof. By Remark 2.11, assigning $g(0)P_j$ to $g(z)z^{\omega_j^\vee} \in G[[z]]z^{\omega_j^\vee} = \mathrm{Gr}_{G, z^{\omega_j^\vee}}$ provides an isomorphism $\mathrm{Gr}_{G, z^{\omega_j^\vee}} \simeq G/P_j$ where $P_j \subseteq G$ is the maximal parabolic subgroup associated to $\omega_j^\vee$.

Furthermore, the affine Springer fibre condition $\mathrm{Ad}_{z^{-\omega_j^\vee} g(z)^{-1}}(\varphi_1) \in \mathfrak{g}[[z]]$ is equivalent to $\mathrm{Ad}_{g(z)^{-1}}(\varphi_1) \in \mathrm{Ad}_{z^{\omega_j^\vee}}(\mathfrak{g}[[z]])$ which, evaluating at $z = 0$, is equivalent under the previous isomorphism to $\mathrm{Ad}_{g(0)^{-1}}(e) \in \mathfrak{p}_j^{\mathrm{opp}}$ (this is the usual Springer fibre condition in G/P_j). Therefore, the statement is equivalent to e belonging to a unique conjugate of the parabolic subalgebra $\mathfrak{p}_j^{\mathrm{opp}}$, which follows by Lemma 1.1. ∎

Remark 2.15 *Lemma 2.14 shows that if $\varphi_1(0) \in \mathfrak{g}$ is a regular nilpotent element then there is only one possible Hecke transformation whose type is a minuscule cocharacter $\omega_j^\vee$. This is a generalisation of a similar fact that appears when performing Hecke transformations in type A (e.g. [70, Proof of Theorem 4.16]), namely that the matrix*

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

only admits one invariant subspace of a given dimension.

We conclude this section with an example.

Example 2.16 *Consider the canonical uniformising $\mathrm{PGL}_3(\mathbb{C})$-Higgs bundle from Example 1.74: $E = K_C \oplus \mathcal{O} \oplus K_C^{-1}$ and $\varphi = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We take as Cartan subalgebra the diagonal matrices with coordinates (e_1, e_2, e_3) (subject to $e_1+e_2+e_3 = 0$) and as simple roots $\beta_1 := e_2 - e_1$ and $\beta_2 = e_3 - e_2$. A choice of simple root vectors is $X_1 := E_{21}$ and $X_2 := E_{32}$, where E_{ij} denotes the matrix with a one on the entry (i, j) and zeroes elsewhere. This example is of Borel type as the Higgs field has constant value equal to $\beta = X_1 + X_2 \in \mathfrak{g}_1$ and the bundle reduces to G_0 the maximal torus of diagonal matrices. Consider the first fundamental cocharacter, which as a cocharacter of $T \subseteq G$ is:*

$$z^{\omega_1^\vee} = \begin{pmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and as an element of $\mathfrak{t} \subseteq \mathfrak{sl}_3(\mathbb{C})$ it is $\omega_1^\vee = \mathrm{diag}(-2/3, 1/3, 1/3)$. Now, let $z = \exp(t)$ and use the fact that $[\omega_i^\vee, X_j] = \beta_j(\omega_i^\vee)X_j = \delta_{ij}X_j$ to compute:

$$\mathrm{Ad}_{z^{k\omega_i^\vee}}(X_j) = \mathrm{Ad}_{\exp(tk\omega_i^\vee)}(X_j) = \exp(\mathrm{ad}(tk\omega_i^\vee))(X_j) = \begin{cases} z^k X_j & i = j, \\ X_j & i \neq j \end{cases}.$$

Thus, in this case, the choice $\sigma = z^{\omega_1^\vee}$ is not in the space of Hecke transformations since $\sigma^{-1} = z^{-\omega_1^\vee}$ maps the Higgs field to $\frac{1}{z}X_1 + X_2$ which has a pole. However we can take $\sigma = z^{-\omega_1^\vee}$. Then the Hecke modified Higgs bundle is

$$E' = K_C(-c) \oplus \mathcal{O} \oplus K_C^{-1}.$$

Moreover, the new Higgs field in local coordinates around z is, according to the previous calculations, equal to $zX_1 + X_2$. If $s_c \in H^0(\mathcal{O}(c))$ denotes the canonical section, we can globally write it as

$$\varphi' = \begin{pmatrix} 0 & 0 & 0 \\ s_c & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus we see how this approach recovers the one in [70, Section 4.2]. Note that we can always revert the Hecke transformation by taking σ^{-1} in the space of Hecke transformations of (E', φ') with respect to the induced local trivialisation.

Here is an example of Hecke transformation that produces a non-fixed nilpotent. We take $\sigma^{-1} = \gamma_+^{-1} z^{\omega_1^\vee}$ where γ_+ is the following element of the root subgroup:

$$\gamma_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $z^{-\omega_1^\vee}$ sends $X_1 \mapsto zX_1$, $X_2 \mapsto X_2$, $X_3 \mapsto zX_3$ and γ_+ sends $X_1 \mapsto X_1 + X_3$, $X_2 \mapsto X_2$. So in the Hecke transformation, locally (on C_1) the Higgs field has value $zX_1 + zX_3 + X_2$ which is nilpotent but not graded (also G_0 is not preserved by this σ).

2.2 The $\mathbb{C}^\times$-action on the space of Hecke transformations at a fixed point

The strategy for the classification of very stable points of Borel type is to introduce a $\mathbb{C}^\times$-action on the space of Hecke transformations of $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ at a point $c \in C$, in such a way that it intertwines with the $\mathbb{C}^\times$-action on $\mathcal{M}(G)$. Now we describe this action for **any** type of fixed point (E, φ) (for any Vinberg $\mathbb{C}^\times$-pair $(G_0, \mathfrak{g}_1)$ with grading element $\zeta \in \mathfrak{g}_0$ and corresponding cocharacter $\xi_\lambda \in G_{ad}$) We keep the same notation of the previous section, in particular we recall that Hecke transformations are performed at $c \in C$, C_1 is a formal disk around c and $C_0 = C \smallsetminus \{c\}$.

In order to cleanly describe our $\mathbb{C}^\times$-action, we need to restrict to local trivialisations which are compatible with the $(G_0, \mathfrak{g}_1)$-reduction.

Definition 2.17 *Let $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ be a G-Higgs bundle fixed by the $\mathbb{C}^\times$-action reducing to a Vinberg pair $(G_0, \mathfrak{g}_1)$, and let $c \in C$. A local trivialisation $t_1 : E_1 \to C_1 \times G$ is called a G_0-**trivialisation** if $t_1(E_{G_0}) = C_1 \times G_0$. If, furthermore, we have $\varphi_1 \in \mathfrak{g}_1[[z]]$, then it is called a $(G_0, \mathfrak{g}_1)$-**trivialisation**.*

Since every principal bundle (for any affine algebraic group, or more generally, any smooth affine group scheme) can be trivialised over a formal disk C_1, it is clear that the previous trivialisations always exist.

Proposition 2.18. *Let (E, φ) be a fixed point of the $\mathbb{C}^\times$-action and fix a local $(G_0, \mathfrak{g}_1)$-trivialisation over C_1 so that the Hecke transformation space is identified with the affine Springer fibre:*

$$\mathrm{Gr}_G^{\varphi_1} = \{\sigma \in \mathrm{Gr}_G : \mathrm{Ad}_{\sigma^{-1}}(\varphi_1) \in \mathfrak{g}[[z]]\} \subseteq \mathrm{Gr}_G .$$

Then, there is a natural $\mathbb{C}^\times$-action in the affine Grassmannian Gr_G:

$$\sigma \mapsto \xi_\lambda^{-1} \sigma,$$

which restricts to the space of Hecke transformations. Denoting by (E', φ') the Hecke transformation by σ and by $(E'_\lambda, \varphi'_\lambda)$ the Hecke transformation by $\xi_\lambda^{-1}\sigma$, we have

$$(E', \lambda\varphi') \simeq (E'_\lambda, \varphi'_\lambda).$$

Proof. It is clear that the action is well defined on the space of Hecke transformations, since $\operatorname{Ad}\sigma^{-1}\xi_\lambda(\varphi_1) = \lambda \cdot \operatorname{Ad}_{\sigma^{-1}}(\varphi_1)$. Note also that, since $Z(G) \subseteq G[[z]]$, the action of an element $\xi_\lambda^{-1} \in G_{\mathrm{ad}}$ is well defined on Gr_G. For the subsequent proof any lift of the element ξ_λ to G can be selected.

The isomorphism

$$(E, \varphi) \simeq (E, \lambda\varphi)$$

can be expressed using local data over C_0 and C_1, $(E_0, C_1 \times G, \Phi, \varphi_0, \varphi_1)$ and $(E_0, C_1 \times G, \Phi, \lambda\varphi_0, \lambda\varphi_1)$ by (f_λ, ξ_λ) where $f_\lambda : E_0 \xrightarrow{\sim} E_0$ is the restriction of the isomorphism to C_0. The compatibility conditions with the Higgs field are

$$(\lambda\varphi_0) \circ f_\lambda = \varphi_0$$

$$(\lambda\varphi_1) \circ \xi_\lambda = \varphi_1$$

(the last one holds automatically due to G-equivariance and the fact that $\operatorname{Ad}_{\xi_\lambda^{-1}}(X) = \lambda^{-1}X$ for any $X \in \mathfrak{g}_1$). The glueing condition is

$$f_\lambda \circ \Phi = \Phi \circ \xi_\lambda$$

over C_{01}.

Now we perform two different operations. On one hand we apply a Hecke transformation by $\sigma \in G((z))$ and then we apply the $\mathbb{C}^\times$-action, yielding

$$(E_0, C_1 \times G, \Phi \circ \sigma, \lambda\varphi_0, \lambda \operatorname{Ad}_{\sigma^{-1}}(\varphi_1)).$$

On the other hand we apply a Hecke transformation by $\xi_\lambda^{-1}\sigma$, yielding

$$(E_0, C_1 \times G, \Phi \circ \xi_\lambda^{-1}\sigma, \varphi_0, \lambda \operatorname{Ad}_{\sigma^{-1}}(\varphi_1)).$$

We claim that both are isomorphic, using the pair $(f_\lambda^{-1}, \mathrm{Id})$. First we check the glueing condition, which is

$$f_\lambda^{-1}\Phi\sigma = \Phi \circ \xi_\lambda^{-1}\sigma.$$

This is equivalent to $f_\lambda \circ \Phi = \Phi \circ \xi_\lambda$ which we know from above is true. Then we check the compatibility with the Higgs fields:

$$\varphi_0 \circ f_\lambda^{-1} = \lambda\varphi_0$$

$$\lambda \operatorname{Ad}_{\sigma^{-1}}(\varphi_1) \circ \mathrm{Id} = \lambda \operatorname{Ad}_{\sigma^{-1}}(\varphi_1).$$

The second is tautological and the first holds as seen above. □

Now restrict to the Borel grading $(G_0, \mathfrak{g}_1)$ with $\zeta = \sum_{j=1}^r \omega_j^\vee$. We show that the curves in the affine Grassmannian from Lemma 2.5 are invariant under this $\mathbb{C}^\times$-action. Therefore, they produce $\mathbb{C}^\times$-orbits in $\mathcal{M}(G)$ via Hecke transformations.

Proposition 2.19. *Let $\mu \in X_*(T)$ be a cocharacter and $\beta^\vee \in \Delta_+^\vee$ a positive coroot with corresponding positive root $\beta \in \Phi^+$. Assume that $m := \beta(\mu) - 1 \geqslant 0$. There exists an element*

$$\sigma_{\mu,\beta} := Li_\beta \begin{pmatrix} 1 & z^m \\ 0 & 1 \end{pmatrix} z^\mu \in \mathrm{Gr}_G$$

such that, for every $\lambda \in \mathbb{C}^\times$, $\xi_\lambda^{-1}\sigma_{\mu,\beta}$ is contained in Gr_μ and has limits $z^{\mu-\beta^\vee}$ when $\lambda \to 0$ and z^μ when $\lambda \to \infty$.

Proof. Since $\mathrm{Ad}_{\xi_\lambda}$ acts with weight $\mathrm{ht}(\beta)$ on X_β, the adjoint action of ξ_λ^{-1} maps $X_\beta \mapsto \lambda^{-\mathrm{ht}(\beta)} X_\beta$. This implies that we have the following curve in Gr_G:

$$\lambda \mapsto \xi_\lambda^{-1} Li_\beta \begin{pmatrix} 1 & z^m \\ 0 & 1 \end{pmatrix} z^\mu = \xi_\lambda^{-1} Li_\beta \begin{pmatrix} 1 & z^m \\ 0 & 1 \end{pmatrix} z^\mu \xi_\lambda = Li_\beta \begin{pmatrix} 1 & \lambda^{-\mathrm{ht}(\beta)} z^m \\ 0 & 1 \end{pmatrix} z^\mu.$$

The result now follows directly from Lemma 2.5 and its proof, noticing that the curve given by $(\sigma_{\mu,\beta})_\lambda$ for $\lambda \in \mathbb{C}^\times$ coincides with $C_{\mu,\beta} \smallsetminus \{z^\mu, z^{\mu-\beta}\}$. □

2.3 Classification of very stable Higgs bundles in terms of the multiplicity divisor

In this section we classify, depending on the multiplicity divisor $\mu(E,\varphi)$ from Definition 1.45, which fixed points (E,φ) of Borel type are very stable. First, we consider Hecke transformations fixed by the $\mathbb{C}^\times$-action of Proposition 2.18, namely, elements of $X_*(T) \subseteq \mathrm{Gr}_G$, which therefore produce another fixed point.

Proposition 2.20. *Let (E,φ) be a $\mathbb{C}^\times$-fixed point of Borel type with multiplicity divisor $\mu(E,\varphi)$ and fix a local $(G_0,\mathfrak{g}_1)$-trivialisation at $c \in C$. Let $\mu \in X_*(T)$. Suppose that $\mu_c(E,\varphi) - \mu \in P^+_{\Delta^\vee}$. Then $z^\mu \in \mathrm{Gr}_G^{\varphi_1}$, and the Hecke transformation by z^μ is another $\mathbb{C}^\times$-fixed point of Borel type with multiplicity divisor $\mu(E,\varphi) - \mu \cdot c$.*

Proof. By definition of the multiplicity divisor we can write $\varphi_1 = \sum_i z^{\beta_i(\mu_c(E,\varphi))} u_i(z) X_i$, where $X_i \in \mathfrak{g}_{\beta_i}$ are simple root vectors and $u_i(z)$ are units in $\mathbb{C}[[z]]$. Now, write $z = \exp(t)$ and use the fact that $[\omega_i^\vee, X_j] = \beta_j(\omega_i^\vee) X_j = \delta_{ij} X_j$ to compute:

$$\mathrm{Ad}_{z^{k\omega_i^\vee}}(X_j) = \mathrm{Ad}_{\exp(tk\omega_i^\vee)}(X_j) = \exp(\mathrm{ad}(tk\omega_i^\vee))(X_j) = \begin{cases} z^k X_j & i = j, \\ X_j & i \neq j \end{cases}.$$

Then, we have

$$\mathrm{Ad}_{z^{-\mu}}(\varphi_1) = \sum_i z^{\beta_i(\mu_c(E,\varphi)-\mu)} u_i(z) X_i,$$

as desired. □

We will also need to know when can we perform Hecke transformations by the curve in Proposition 2.19, in other words, when is said curve fully contained in the relevant affine Springer fibre.

Lemma 2.21. *Let (E,φ) be a $\mathbb{C}^\times$-fixed point of Borel type with multiplicity divisor $\mu(E,\varphi)$ and fix a local $(G_0,\mathfrak{g}_1)$-trivialisation at $c \in C$. Let $\mu \in X_*(T)$ and $\beta \in \Delta^+$. Then, $\sigma_{\mu,\beta} \in \mathrm{Gr}_G^{\varphi_1}$ if and only if for all $i \in \{1,\dots,r\}$ and $q_i \geqslant 0$ such that $\beta_i + q_i\beta \in \Delta$, we have*

$$\beta_i(\mu_c(E,\varphi) - \mu) - q_i \geqslant 0.$$

Proof. Set $m = \beta(\mu) - 1$ and $\nu := \exp(z^m X_\beta)$ where X_β is a nonzero root vector for β. We have to check that $\mathrm{Ad}_{z^{-\mu}\nu^{-1}} \varphi_1 \in \mathfrak{g}[[z]]$. We have that $\varphi_1 = \sum_i z^{\beta_i(\mu_c(E,\varphi))} u_i(z) X_i$ with $u_i(z)$ units in $\mathbb{C}[[z]]$. Using the fact that, whenever $\beta + \beta'$ is a root, we have $[X_\beta, X_{\beta'}] \in \langle X_{\beta+\beta'} \rangle$, we get:

$$\mathrm{Ad}_{\nu^{-1}} \varphi_1 = \sum_{q \geqslant 0} z^{qm} \varphi^{q\beta}.$$

Here

$$\varphi^{q\beta} := \sum_{i=1}^{r} D_{i,q}(z) z^{\beta_i(\mu_c(E,\varphi))} X_{\beta_i+q\beta},$$

where the $D_{i,q}(z)$ are units in $\mathbb{C}[[z]]$ and with the convention that $X_{\beta_i+q\beta} = 0$ when $\beta_i + q\beta$ is not a root. Notice that the $\varphi^{q\beta}$ are linearly independent from each other because the vectors appearing in them correspond to roots of height $\mathrm{ht}(q\beta)+1$. Hence $\sigma_{\mu,\beta} \in \mathrm{Gr}_G^{\varphi_1}$ if and only if $\mathrm{Ad}_{z^{-\mu}} z^{qm}\varphi^{q\beta} \in \mathfrak{g}[[z]]$ for all $q \geqslant 0$. Now, using that $\beta(\mu) = m + 1$, we get:

$$\begin{aligned}\mathrm{Ad}_{z^{-\mu}} z^{qm}\varphi^{q\beta} &= \sum_{i=1}^{r} D_{i,q}(z) z^{qm} z^{\beta_i(\mu_c(E,\varphi))} z^{-\beta_i(\mu)-q\beta(\mu)} X_{\beta_i+q\beta} \\ &= \sum_{i=1}^{r} D_{i,q}(z) z^{\beta_i(\mu_c(E,\varphi))-\beta_i(\mu)-q} X_{\beta_i+q\beta},\end{aligned}$$

and the lemma follows. □

From the previous lemma it follows that not every choice of $\beta \in \Delta^+$ such that $\mu_c(E,\varphi) - \beta^\vee$ is still dominant allows to perform a curve of Hecke transformations of the previous kind, interpolating between (E,φ) and a point whose multiplicity divisor at c is $\mu_c(E,\varphi) - \beta^\vee$. However, it turns out that if $\mu_c(E,\varphi) - \beta^\vee \leqslant \mu_c(E,\varphi)$ is a covering relation (i.e. there are no other intermediate dominant coweights) in the Bruhat order, the condition of the previous lemma is always satisfied. Therefore, the following corollary can be deduced.

Corollary 2.22. *Let (E,φ) be a $\mathbb{C}^\times$-fixed point of Borel type with multiplicity divisor $\mu(E,\varphi)$ and fix a local $(G_0,\mathfrak{g}_1)$-trivialisation at $c \in C$. Suppose that $\mu_c(E,\varphi)$ is not minuscule. Then, there exists a positive root $\beta \in \Delta^+$ such that $\mu_c(E,\varphi) - \beta^\vee \in P^+_{\Delta^\vee}$ and $\sigma_{\beta^\vee,\beta} \in \mathrm{Gr}_G^{\varphi_1}$.*

Proof. Since $\mu_c(E,\varphi)$ is not minuscule, we can find a dominant coweight $\nu \in X_*^+(T)$ such that $\nu < \mu_c(E,\varphi)$ in the Bruhat order. Assume that the previous is a covering relation, that is, there is no intermediate dominant coweight. Then, [119, Corollary 2.7] shows that $\nu = \mu_c(E,\varphi) - \beta^\vee$ for some $\beta \in \Delta^+$, and according to Lemma 2.21, it suffices to prove for all $\beta_i \in \Pi$ that

$$\beta_i(\mu_c(E,\varphi) - \beta^\vee) \geqslant q_i$$

where q_i is the greatest integer such that $\beta_i + q_i\beta \in \Delta$.

Let $\Pi^\vee = \{\beta_1^\vee, \ldots, \beta_r^\vee\}$ denote the simple root system for $\Delta^\vee$ which consists of the coroots of the simple roots. Let $\Pi^\vee_{\beta^\vee} \subseteq \Pi^\vee$ denote the support of $\beta^\vee$, that is, the simple coroots appearing with non-zero coefficient in the expression of $\beta^\vee$ in terms of $\Pi^\vee$. Let $\Delta^\vee_{\beta^\vee} \subseteq \Delta^\vee$ denote the root subsystem spanned by $\Pi^\vee_{\beta^\vee}$. Since $\nu < \mu_c(E,\varphi)$ is a covering relation, [119, Theorem 2.6 and Theorem 2.8] shows that $\Delta^\vee_{\beta^\vee}$ is irreducible and $\beta^\vee$ is either the highest short root of $\Delta^\vee_{\beta^\vee}$, or $\Delta^\vee_{\beta^\vee} \simeq G_2$ and $\beta^\vee$ is the sum of the two roots in $\Pi^\vee_{\beta^\vee}$. We now check the condition in the three possible cases for a given $\beta_i \in \Pi$:

- If $\beta_i^\vee \notin \Pi^\vee_{\beta^\vee}$, then $\beta_i - \beta \notin \Delta$. In other words, the root string through β_i in the direction of β starts at β_i, which implies
$$q_i = -\beta_i(\beta^\vee)$$
and the condition follows since $\beta_i(\mu_c(E,\varphi)) \geqslant 0$ as $\mu_c(E,\varphi)$ is dominant.
- If $\beta_i^\vee \in \Pi^\vee_{\beta^\vee}$ and $\beta^\vee$ equals the highest short root of $\Delta^\vee_{\beta^\vee}$, then β is the highest root of $\Delta_\beta \subseteq \Delta$, the root subsystem spanned by the support of β. Therefore, $\beta_i + \beta \notin \Delta$ and $q_i = 0$, so the condition follows from the dominance of $\mu_c(E,\varphi) - \beta^\vee$.

- If $\beta_i^\vee \in \Pi_{\beta^\vee}^\vee$ and $\beta^\vee = \beta_i^\vee + \beta_k^\vee$ where $\Pi_{\beta^\vee}^\vee = \{\beta_i^\vee, \beta_k^\vee\}$ and $\Delta_{\beta^\vee}^\vee \simeq G_2$, we distinguish two cases. If $\beta_i^\vee$ is a long root in the copy of G_2, then $\beta = (\beta_i^\vee + \beta_k^\vee)^\vee = 3\beta_i + \beta_k$ and $q_i = 0$, so the condition follows from the dominance of $\mu_c(E,\varphi) - \beta^\vee$. On the other hand, if $\beta_i^\vee$ is a short root in the copy of G_2, then $\beta = \beta_i + 3\beta_k$ so that $q_i = 1$. Therefore, we need to check whether

  $$1 \leqslant \beta_i(\mu_c(E,\varphi) - \beta^\vee) = \beta_i(\mu_c(E,\varphi)) - \beta_i(\beta_i^\vee + \beta_k^\vee) = \beta_i(\mu_c(E,\varphi)) - 2 - (-3) = \beta_i(\mu_c(E,\varphi)) + 1$$

  which follows from the dominance of $\mu_c(E,\varphi)$.

□

Remark 2.23 *Note that in the previous proof the choice of a root $\beta \in \Delta^+$ given by a covering relation is sufficient, but other roots may work as well depending on the specific root system and $\mu_c(E,\varphi)$. For example, if $\mathfrak{g}$ is simply laced, any $\beta \in \Delta^+$ such that $\mu_c(E,\varphi) - \beta^\vee$ is dominant satisfies that $\sigma_{\beta^\vee,\beta} \in \mathrm{Gr}_G^{\varphi_1}$, since the assumptions imply that we always have $q_i = 0$: otherwise, $\beta_i + \beta$ would be a root and applying the symmetry s_β together with the dominance assumptions would imply that $\beta_i - \beta$ would also be a root, yet it is not possible that $\beta_i - \beta, \beta_i, \beta_i + \beta$ are all roots in a simply laced root system.*

We are finally in position of proving the main result of this chapter. Recall from Definition 1.48 the notion of a *reduced* multiplicity divisor.

Theorem 2.24. *Let (E,φ) be a smooth $\mathbb{C}^\times$-fixed point of Borel type with multiplicity divisor $\mu(E,\varphi)$. Then, (E,φ) is very stable if and only if $\mu(E,\varphi)$ is reduced.*

Proof. Suppose that some $\mu_c(E,\varphi) =: \mu$ is not minuscule and fix a local $(G_0,\mathfrak{g}_1)$-trivialisation at $c \in C$. We can use Corollary 2.22 to find a coroot $\beta^\vee$ with $\mu - \beta^\vee$ dominant and such that $\sigma_{\beta^\vee,\beta} \in \mathrm{Gr}_G^{\varphi_1}$. Then, by Proposition 2.19 the expression $\xi_\lambda^{-1}\sigma_{\beta^\vee,\beta}$ defines a $\mathbb{C}^\times$-orbit in $\mathrm{Gr}_G^{\varphi_1}$ whose limit points are 0 at $\lambda \to 0$ and $\beta^\vee$ at $\lambda \to \infty$, in such a way that the Hecke transformations of (E,φ) given by each $\xi_\lambda^{-1}\sigma_{\beta^\vee,\beta}$ result in a $\mathbb{C}^\times$-orbit of G-Higgs bundles. By Proposition 2.20, said orbit limits to (E,φ) when $\lambda \to 0$ and (E',φ') when $\lambda \to \infty$, where (E',φ') is a $(G_0,\mathfrak{g}_1)$-Higgs pair with multiplicity divisor $\mu(E',\varphi') = \mu(E,\varphi) - \beta^\vee \cdot c$. By Corollary 1.51 (E',φ') is stable. By openness of stability, the whole $\mathbb{P}^1_{\mathbb{C}}$ is stable and thus contained in $\mathcal{M}(G)$, implying that (E,φ) is wobbly as its upward flow is not closed.

For the converse, notice that we may assume that $G = G_{\mathrm{ad}}$ is of adjoint type. Otherwise, we may use the map $\mathcal{M}(G) \to \mathcal{M}(G_{\mathrm{ad}})$ given by extension of structure group via the projection $G \to G_{\mathrm{ad}}$. If (E,φ) were wobbly, its image in $\mathcal{M}(G_{\mathrm{ad}})$ would also be wobbly with the same multiplicities, a contradiction.

Therefore, we suppose that $G = G_{\mathrm{ad}}$. In particular, the fundamental coweights satisfy $\omega_j^\vee \in X_*(T)$. We proceed by induction on $N(E,\varphi) := \sum_{c\in C}\sum_{i=1}^r \beta_i(\mu_c(E,\varphi))$. The case $N(E,\varphi) = 0$ corresponds to $\mu(E,\varphi) = 0$, i.e. that of Example 1.74, which was seen to be very stable as a consequence of the upward flow being a section of the Hitchin map. Now suppose that there is some nonzero $\mu_c(E,\varphi) =: \mu$. It is minuscule, in particular of the form $\mu = \omega_j^\vee$. Now assume that (E',φ') is a nilpotent Higgs bundle on the upward flow of (E,φ). By Proposition 1.71, there is a reduction $\sigma \in H^0(C, E'(G/P_{\leqslant 0}))$ to a $(P_{\leqslant 0}, \mathfrak{p}_{\leqslant 0} \oplus \mathfrak{g}_1)$-Higgs pair such that $\mathrm{Gr}_\sigma(E',\varphi') \simeq (E,\varphi)$. The existence of this reduction allows to choose a $(P_{\leqslant 0}, \mathfrak{p}_{\leqslant 0} \oplus \mathfrak{g}_1)$-trivialisation around $c \in C$, meaning (with the notation of Section 2.1) an isomorphism $t_1 : E_1 \xrightarrow{\sim} C_1 \times G$ such that $t_1(E_\sigma) = C_1 \times P_{\leqslant 0}$ and $\varphi_1 \in (\mathfrak{p}_{\leqslant 0} \oplus \mathfrak{g}_1)[[z]]$. For $k \leqslant 1$, let $\varphi_{1,k} \in \mathfrak{g}_k[[z]]$ denote the projection to $\mathfrak{g}_k$ of the local expression. Moreover, taking associated graded results in an induced $(G_0,\mathfrak{g}_1)$-trivialisation of $\mathrm{Gr}_\sigma(E',\varphi')$ around $c \in C$, with the property that $Gr_\sigma(\varphi')_1 = \varphi'_{1,1} \in \mathfrak{g}_1[[z]]$.

Using that $\mathrm{Gr}_\sigma(E',\varphi') \simeq (E,\varphi)$, Proposition 2.20 shows that $z^{\omega_j^\vee} \in \mathrm{Gr}_G^{\varphi'_{1,1}}$. In particular we have $z^{\omega_j^\vee} \in \mathrm{Gr}_G^{\varphi'_1}$, since the fact that $\beta(-\omega_j^\vee) \geqslant 0$ for any negative root $\beta \in \Delta^-$ implies that $\mathrm{Ad}_{z^{-\omega_j^\vee}}(\varphi'_{1,k}) \in \mathfrak{g}[[z]]$ for any $k \leqslant 0$. Thus, we have shown that (E',φ') admits a Hecke transformation by $z^{\omega_j^\vee}$ which we denote by (F',θ'). Since the Hecke transformation is local around c and $G/P_{\leqslant 0}$ is proper, (F',θ') is naturally endowed with a reduction to $P_{\leqslant 0}$ which, by Proposition 1.71 (using Corollary 1.52 to show that (F,θ) is stable), implies that (F',θ') is in the upward flow of the Hecke transformation of $\mathrm{Gr}_\sigma(E',\varphi')$ by $z^{\omega_j^\vee}$, denoted by (F,θ).

As explained above, Corollary 1.52 shows that (F,θ) is stable. Moreover, by Propositions 1.49 and 2.20 it defines a smooth $\mathbb{C}^\times$-fixed point in $\mathcal{M}(G)$ with multiplicity divisor $\mu(F,\theta) = \mu(E,\varphi) - \omega_j^\vee \cdot c$. Therefore, $N(F,\theta) = N(E,\varphi) - 1$ and (F,θ) is very stable by induction. Thus, since (F',θ') is also nilpotent (as Hecke transformations preserve Hitchin fibres) we must have $(F',\theta') \simeq (F,\theta)$.

Finally, note that by construction there is a local trivialisation of (F',θ') around c such that the Hecke transformation by $z^{-\omega_j^\vee}$ recovers (E',φ'). Due to the isomorphism $(F',\theta') \simeq (F,\theta)$, there must be an element $\eta \in \mathrm{Gr}^{\theta_1}_{G,-\omega_j^\vee} = G[[z]]z^{-\omega_j^\vee} \cap \mathrm{Gr}_G^{\theta_1}$ that corresponds to this $z^{-\omega_j^\vee}$, since the type of the Hecke transformation does not depend on the chosen trivialisation. But, since $\omega_j^\vee$ is minuscule (thus so is $-\omega_j^\vee$) and $\theta_1(0)$ is regular nilpotent (because $\mu_c(F,\theta) = \mu_c(E,\varphi) - \omega_j^\vee = 0$), Lemma 2.14 implies that $\eta = z^{-\omega_j^\vee}$, which we know is in $\mathrm{Gr}_G^{\theta_1}$ as its Hecke transformation recovers $\mathrm{Gr}_\sigma(E,\varphi)$. Therefore, we deduce

$$(E',\varphi') \simeq \mathrm{Gr}_\sigma(E',\varphi') \simeq (E,\varphi)$$

and (E,φ) is very stable. □

We finish by illustrating the classification theorem for classical simple groups. We follow the numbering of the simple roots from the tables in [85, Appendix C].

Example 2.25 *If $G = \mathrm{SL}_n(\mathbb{C})$ or $G = \mathrm{PGL}_n(\mathbb{C})$, recall from Example 1.40 that a fixed point of Borel type consists of n line bundles $(L_1,\dots,L_n)$ over C (with trivial product in the case of $\mathrm{SL}_n(\mathbb{C})$, and up to tensoring by the same line bundle for $\mathrm{PGL}_n(\mathbb{C})$) and $n-1$ nonzero sections $\varphi_i \in H^0(C, L_i^* L_{i+1} K_C)$. Since the minuscule coweights for type A_{n-1} are $\{0,\omega_1^\vee,\dots,\omega_{n-1}^\vee\}$, a stable fixed point of Borel type is very stable if and only if the divisor $\varphi_1 + \cdots + \varphi_{n-1}$ is reduced, recovering [70, Theorem 4.16].*

Example 2.26 *For $G = \mathrm{SO}_{2n+1}(\mathbb{C})$, recall from Example 1.42 that fixed points of Borel type are given by n line bundles $(L_1,\dots,L_n)$ over C, $n-1$ nonzero sections $\varphi_i \in H^0(C, L_i^* L_{i+1} K_C)$ and a nonzero section $\varphi_n \in H^0(C, L_n^* K_C)$. Minuscule coweights in type B_n (corresponding to minuscule weights in type C_n) are $\{0,\omega_1^\vee\}$, so a smooth fixed point of Borel type is very stable if and only if the divisor $\varphi_1 + 2\varphi_2 + \cdots + 2\varphi_{n-1} + 2\varphi_n$ is reduced.*

Example 2.27 *In the case of $G = \mathrm{Sp}_{2n}(\mathbb{C})$, fixed points of Borel type are given by n line bundles $(L_1,\dots,L_n)$ over C, $n-1$ nonzero sections $\varphi_i \in H^0(C, L_i^* L_{i+1} K_C)$ and a nonzero section $\varphi_n \in H^0(C, (L_n^2)^* K_C)$, as was seen in Example 1.43. Therefore, since minuscule coweights in type C_n are $\{0,\omega_n^\vee\}$, we have that a smooth fixed point of Borel type is very stable if and only if the divisor $2\varphi_1 + \cdots + 2\varphi_{n-1} + \varphi_n$ is reduced.*

Example 2.28 *For $G = \mathrm{SO}_{2n}(\mathbb{C})$, Example 1.41 shows that fixed points of Borel type are given by n line bundles $(L_1,\dots,L_n)$ over C, $n-1$ nonzero sections $\varphi_i \in H^0(C, L_i^* L_{i+1} K_C)$ and a nonzero section $\varphi_n \in H^0(C, L_{n-1}^* L_n^* K_C)$. Minuscule coweights in type D_n are $\{0,\omega_1^\vee,\omega_{n-1}^\vee,\omega_n^\vee\}$, so that a smooth fixed point of Borel type is very stable if and only if the divisor $\varphi_1 + 2\varphi_2 + \cdots + 2\varphi_{n-2} + \varphi_{n-1} + \varphi_n$ is reduced.*

Remark 2.29 *We can use Remark 1.46 to get restrictions on which fixed point components of Borel type can contain very stable points. Indeed, recalling the topological type* $d_E = \deg(\mu(E,\varphi)) + \mu_{\text{can}}$ *from said remark, together with Theorem 2.24, we get that if a fixed point component contains very stable points then* $d_E \in \pi_1(G_0) \simeq X_*(T)$ *must verify that* $d_E - \mu_{can}$ *can be written as a sum of minuscule coweights in* $\mathfrak{t}$*. In the case of* $G = \mathrm{PGL}_n(\mathbb{C})$*, this imposes no restriction as all fundamental coweights are minuscule (indeed, in this case there are very stable points in every component, see [70, Corollary 4.19]). In any other type, however, this restricts which components are possible. In particular, in types without nonzero minuscule coweights* (E_8, F_4, G_2) *only components containing everywhere regular Higgs bundles may contain very stable points.*

2.4 Virtual equivariant multiplicities and Dynkin polynomials

We start by recalling the following notion from [70, Definition 5.3].

Definition 2.30 *Let* $\mathcal{E} := (E,\varphi) \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ *be a smooth fixed point of the* $\mathbb{C}^\times$*-action. We define its* ***virtual equivariant multiplicity*** *as*

$$m_{\mathcal{E}}(t) := \frac{\chi(\mathrm{Sym}(T^{+*}_{\mathcal{E}}))}{\chi(\mathrm{Sym}(\mathcal{A}(G)^*))} \in \mathbb{Z}((t)),$$

where $T^+_{\mathcal{E}} := T^+_{\mathcal{E}}\mathcal{M}(G)$ *is the subspace of positive weights (see Section 1.5) of the tangent* $T_{\mathcal{E}}\mathcal{M}(G)$*,* $\mathcal{A}(G)$ *is the Hitchin base and* χ *denotes the character of a* $\mathbb{C}^\times$*-representation, that is, the Laurent series where the coefficient of* t^{-k} *is the dimension of the* k*-th weight subspace.*

Virtual equivariant multiplicities are interesting objects to associate to a fixed point due to the following properties.

Proposition 2.31 ([70, Theorem 5.2, Corollary 5.4]). *Let* $\mathcal{E} \in \mathcal{M}(G)^{s\mathbb{C}^\times}$ *be a very stable smooth fixed point. Then,* $m_{\mathcal{E}}(t)$ *is a monic, palindromic polynomial with non-negative integer coefficients such that* $m_{\mathcal{E}}(1)$ *equals the multiplicity of the fixed point component containing* $\mathcal{E}$ *in the nilpotent cone* $h^{-1}(0)$*.*

The goal of this section is to compute $m_{\mathcal{E}}(t)$ for the fixed points of Borel type in terms of the Lie-theoretic data of $\mathfrak{g}$ and then show that in the very stable cases they agree with the Dynkin polynomials of the corresponding highest weight representation of the Langlands dual group $G^\vee$. This computation was outlined in [70, Section 8.1] and we now show how it can be carried out within our framework.

For a $\mathbb{C}^\times$-representation with positive weights $V = \bigoplus_{\lambda>0} V_\lambda$, we have

$$\chi(\mathrm{Sym}(V^*)) = \prod_{\lambda>0} \frac{1}{(1-t^\lambda)^{\dim V_\lambda}}.$$

It now suffices to compute the dimensions of the weight spaces in both $T^+_{\mathcal{E}}\mathcal{M}(G)$ and $\mathcal{A}(G)$.

We can start by computing the denominator $\chi(\mathrm{Sym}(\mathcal{A}^*(G)))$. Recall that if $\{e_j\}_{j=1}^r$ are the exponents of $\mathfrak{g}$, then $\mathcal{A}(G) = \bigoplus_{j=1}^r H^0(X, K_C^{e_j+1})$ and $\mathbb{C}^\times$ acts with weight e_j+1 on the j-th summand. Thus

$$\chi(\mathrm{Sym}(\mathcal{A}^*(G))) = \prod_{j=1}^r \frac{1}{(1-t^{e_j+1})^{(2e_j+1)(g-1)}}. \tag{2.4.1}$$

Now we compute $\chi(\mathrm{Sym}(T^{+*}_{\mathcal{E}}))$. Recall from the proof of Proposition 1.71 that the subspace of $T_{\mathcal{E}}$ with weight $-j$ is $\mathbb{H}^1(E(\mathfrak{g}_j) \xrightarrow{[\varphi,-]} E(\mathfrak{g}_{j+1}) \otimes K_C)$, so it will be necessary to compute the Euler characteristics $\chi(E(\mathfrak{g}_j))$ and, in turn, $\deg E(\mathfrak{g}_j)$ for the different $\mathfrak{g}_j$.

The degree of a bundle associated to a representation $\rho : G_0 \to \mathrm{GL}(\mathfrak{g}_j)$ can be computed by means of the character $\chi_\rho : G_0 \to \mathbb{C}^\times$ given by $\chi_\rho(g_0) := \det(\rho(g_0))$. From the structure of the Borel grading, we have:

$$\chi_\rho(g_0) = \prod_{\beta\in\Delta,\mathrm{ht}(\beta)=j} \beta(g_0).$$

Let $\mu(E,\varphi)$ be the multiplicity divisor of (E,φ) and $\mu := \deg(\mu(E,\varphi)) = \sum_{c\in C} \mu_c(E,\varphi) \in X_*(T)$. Then, by Remark 1.46, we have that the topological type of E in $\pi_1(G_0)$ is given by $\mu + \mu_{can}$ where $\mu_{can} = -(2g-2)\sum_{i=1}^r \omega_i^\vee$ is the topological type of the canonical uniformising Higgs bundle.

Hence,

$$d_{-j}(\mu) := \deg E(\mathfrak{g}_j) = \sum_{\beta\in\Delta,\mathrm{ht}(\beta)=j} (\beta(\mu) + \beta(\mu_{can})) = -jr_{-j}(2g-2) + \sum_{\beta\in\Delta,\mathrm{ht}(\beta)=j} \beta(\mu),$$

where

$$r_{-j} := \dim \mathfrak{g}_j = |\{\beta \in \Delta : \mathrm{ht}(\beta) = j\}|.$$

We used that for β a root of height j we have $\beta(\mu_{can}) = -j(2g-2)$. We also define $d_0(\mu) = 0$ since $\deg E(\mathfrak{g}_0) = 0$. By Riemann–Roch we deduce

$$\chi(E(\mathfrak{g}_j)) = d_{-j}(\mu) + r_{-j}(1-g),$$
$$\chi(E(\mathfrak{g}_j) \otimes K_C) = d_{-j}(\mu) + r_{-j}(g-1).$$

Hence for $j > 0$ we have:

$$\dim T^+_{\mathcal{E},j} = \chi(E(\mathfrak{g}_{-j+1}) \otimes K_C) - \chi(E(\mathfrak{g}_{-j})) = d_{j-1}(\mu) - d_j(\mu) + (r_{j-1} + r_j)(g-1). \tag{2.4.2}$$

Using Equations 2.4.1 and 2.4.2, we can compute any desired virtual equivariant multiplicities. Table 2.1 records the cases where μ is minuscule and nonzero. In other words, these are the cases where $\mathcal{E}$ only has a nonzero multiplicity vector at a single point $c \in C$ and said nonzero vector corresponds to a minuscule coweight. The numbering for the fundamental coweights follows the tables in [85, Appendix C].

A direct comparison with [105, Table 1] shows the agreement of the virtual equivariant multiplicity in the above cases with the *Dynkin polynomial* associated to a dominant coweight $\mu \in X^+_*(T)$, which is given by

$$\mathcal{D}_\mu(t) := \prod_{\beta\in\Delta^+} \frac{1 - t^{\beta(\rho^\vee+\mu)}}{1 - t^{\beta(\rho^\vee)}},$$

where $\rho^\vee := \frac{1}{2}\sum_{\beta\in\Delta^+} \beta^\vee$.

Thus, we can state the following proposition (see also [70, Section 8.1]) computing the virtual equivariant multiplicities for very stable fixed points.

Proposition 2.32. *Let $\omega_i^\vee \in X^+_*(T)$ be a minuscule dominant cocharacter, $c \in C$ be a point and $\mathcal{E} = (E,\varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ be a fixed point with multiplicity divisor $\mu(E,\varphi) = \omega_i^\vee \cdot c$. Then, we have the identity*

Table 2.1 Equivariant multiplicities for minuscule upward flows

Type of $\mathfrak{g}$	μ	$m_{\mathcal{E}}(t)$
A_n	$\omega_i^\vee$	$\prod_{j=1}^{i} \frac{1-t^{n-j+1}}{1-t^j}$
B_n	$\omega_1^\vee$	$1+t+\cdots+t^{2n-1}$
C_n	$\omega_n^\vee$	$\prod_{j=1}^{n}(1+t^j)$
D_n	$\omega_1^\vee$	$(1+t^{n-1})(1+t+\cdots+t^{n-1})$
	$\omega_{n-1}^\vee, \omega_n^\vee$	$\prod_{j=1}^{n}(1+t^j)$
E_6	$\omega_1^\vee, \omega_6^\vee$	$(1+t^4+t^8)(1+t+\cdots+t^8)$
E_7	$\omega_7^\vee$	$(1+t^5)(1+t^9)(1+t+\cdots+t^{13})$

$$m_{\mathcal{E}}(t) = \mathcal{D}_{\omega_i^\vee}(t)$$

between its virtual equivariant multiplicity and the corresponding Dynkin polynomial.

Remark 2.33 *Computing the virtual equivariant multiplicities for sums of non-minuscule fundamental coweights results in rational functions which are not polynomials, providing another indirect proof of one of the directions in Theorem 2.24. This is not always the case: in other fixed point types (e.g. [109]), virtual equivariant multiplicities are often polynomials even in components without very stable points.*

2.5 The Hitchin map restricted to very stable upward flows

Let $\mu \in X_*^+(T)$ be a dominant cocharacter, $c \in C$ and $\mathcal{E}^\mu := (E^\mu, \varphi^\mu)$ be a smooth fixed point of Borel type with $\mu(E^\mu, \varphi^\mu) = \mu \cdot c$. By Proposition 2.20, it can be obtained by applying a Hecke transformation of one of the Higgs bundles $\mathcal{E}^0 := (E^0, \varphi^0)$ from Example 1.74 by the element $z^{-\mu}$ with respect to a local $(G_0, \mathfrak{g}_1)$-trivialisation at $c \in C$. Let

$$W_\mu^+ := W_{\mathcal{E}^\mu}^+$$

be its upward flow. By Theorem 2.24, $\mathcal{E}^\mu$ is very stable if and only if μ is minuscule. In fact, if μ is minuscule, the smoothness follows directly from Propositions 1.49 and 1.50, since $\beta_j(\deg(\mu(E^\mu, \varphi^\mu))) = \beta_j(\mu) \in \{0, 1\}$ so it is always strictly smaller than $2g-2$ for $g \geqslant 2$.

The goal of this section is to give a description of

$$h_\mu := h_G|_{W_\mu^+} : W_\mu^+ \to \mathcal{A}(G)$$

which, when μ is minuscule, is a finite, flat surjective map. We follow the a similar approach to the one by Hausel [67] for $G = \mathrm{GL}_n(\mathbb{C})$, using the techniques developed in this chapter.

For now, we work with any dominant $\mu \in X_*^+(T)$. Up to this point, for the classification of very stable and wobbly Higgs bundles, we have only considered Hecke transformations of fixed points of the $\mathbb{C}^\times$-action. Since Hecke transformations preserve the fibres of the Hitchin map, this means that the result always belonged in $h_G^{-1}(0)$. Naturally, this was enough for distinguishing very stable and wobbly points.

However, in order to describe the whole upward flow, we can perform Hecke transformations at different points $\mathcal{E}_a = (E^0, \varphi^a)$ of $W_{\mathcal{E}^0}^+$ (see Example 1.74). This allows to give an explicit description of h_μ.

Despite not being fixed points of the $\mathbb{C}^\times$-action, we are able to study these Hecke transformations because the principal bundle E^0 in $\mathcal{E}_a$ does not change, and its structure group always reduces to the maximal torus $G_0 = T$. In other words, we exploit the reduction of $\mathcal{E}_a$ to a $(G_0, \mathfrak{g}_1 \oplus \mathfrak{g}_{\leqslant -1})$-Higgs pair. Similar to the case of fixed points, for the Hecke transformations we work with a local $(G_0, \mathfrak{g}_1 \oplus \mathfrak{g}_{\leqslant -1})$-trivialisation at $c \in C$, i.e. one such that $t_1(E_{G_0}) = C_1 \times G_0$ and $\varphi_1^a \in (\mathfrak{g}_1 \oplus \mathfrak{g}_{\leqslant -1})[[z]]$.

Let $\mathcal{S} := e + C_{\mathfrak{g}}(f) = \langle e, f_1, \dots, f_r \rangle \subseteq \mathfrak{g}$ be the Kostant section [89, Theorem 0.10], isomorphic to $\mathfrak{t}/W$, used to construct $\mathcal{E}^0$ in Example 1.74.

Remark 2.34 *The $\mathbb{C}^\times$-action on the Hitchin base making h_G equivariant, i.e. the scaling of $a_i \in H^0(C, K_C^{\deg p_i})$ with weight $\deg(p_i)$, comes from the action on $\mathcal{S}$ with weight zero on e and weight $\deg(p_i)$ on f_i. It can be cleanly described by the map*

$$X \mapsto \lambda \operatorname{Ad}_{\xi_\lambda^{-1}}(X),$$

where $\xi_\bullet : \mathbb{C}^\times \to G_{\mathrm{ad}}$ denotes as usual the cocharacter generated by $\zeta = \sum_{j=1}^r \omega_j^\vee$.

Proposition 2.35. *Let $\mathcal{E}_a = (E^0, \varphi^a) \in W_0^+$ be the point corresponding to $a \in \mathcal{A}(G)$. Fix a local $(G_0, \mathfrak{g}_1 \oplus \mathfrak{g}_{\leqslant -1})$-trivialisation at $c \in C$, identifying the space of Hecke transformations of $\mathcal{E}_a$ with $\mathrm{Gr}_G^{X_a}$ where $X_a := \varphi_1^a(0) \in \mathcal{S}$. Suppose that $\sigma \in \mathrm{Gr}_G^{X_a}$. Then, we have $\xi_\lambda^{-1}\sigma \in \mathrm{Gr}_G^{X_{\lambda a}}$, and the Hecke transformation of $\mathcal{E}_a$ by σ followed by the action of λ is isomorphic to the Hecke transformation of $\mathcal{E}_{\lambda a}$ by $\xi_\lambda^{-1}\sigma$.*

Proof. We start by remarking that since the bundle E^0 does not change across the Hitchin section, the above statement is well posed: fixing one local trivialisation allows us to talk about Hecke transformations for any point in W_0^+ on the same footing.

Having remarked this, note that the proof is completely analogous to that of Proposition 2.18. The only difference is noting that, when performing a Hecke transformation by $\xi_\lambda^{-1}\sigma$ to $\mathcal{E}^{\lambda a}$, we get by Remark 2.34 that

$$\operatorname{Ad}_{\sigma^{-1}\xi_\lambda}(\varphi_1^{\lambda a}) = \operatorname{Ad}_{\sigma^{-1}\xi_\lambda}(\lambda \operatorname{Ad}_{\xi_\lambda^{-1}} \varphi_1^a) = \lambda \operatorname{Ad}_{\sigma^{-1}}(\varphi_1^a).$$

□

Corollary 2.36. *Under the same hypotheses as in Proposition 2.35, and assuming that the involved Hecke transformations are stable, we have by taking limits the Hecke transformation of $\mathcal{E}_a$ by σ belongs to $W^+_{\mathcal{E}^{\sigma_0}}$, where $\mathcal{E}^{\sigma_0}$ is the Hecke transformation of $\mathcal{E}^0$ by $\sigma_0 := \lim_{\lambda \to 0} \xi_\lambda^{-1}\sigma \in X^*(T)$, which exists because Gr_G is ind-projective.*

Note that the stability assumption amounts to the stability of $\mathcal{E}^{\sigma_0}$ by openness and $\mathbb{C}^\times$-invariance of stability. From Corollary 2.36 we can obtain points in W_μ^+ by looking at the upward flow $W^+_{z^{-\mu}}$ of $z^{-\mu}$ inside of $\mathrm{Gr}_G^{\mathcal{S}} := \bigcup_{X \in \mathcal{S}} \mathrm{Gr}_G^X$ under the $\mathbb{C}^\times$-action given by ξ_λ^{-1}.

Now we start to assume that $\mu \in X_*^+(T)$ is minuscule. Recall from Definition 2.3 that $\mathrm{Gr}_{G,-\mu} = G[[z]]z^{-\mu}$ denotes the corresponding Bruhat cell. Consider the elements $X \in \mathcal{S}$ of the Kostant section used to define the Hitchin section, and define the subspace

$$\mathrm{Gr}^{\mathcal{S}}_{G,-\mu} := \bigcup_{X \in \mathcal{S}} \mathrm{Gr}^X_{G,-\mu} = \bigcup_{X \in \mathcal{S}} \{\sigma \in G[[z]]z^{-\mu} : \operatorname{Ad}_{\sigma^{-1}}(X) \in \mathfrak{g}[[z]]\} \subseteq \mathrm{Gr}_{G,-\mu} \subseteq \mathrm{Gr}_G$$

Consider the parabolic subgroup $P_{-\mu} \subseteq G$ given by $-\mu$ (as explained in Section 1.1) and the partial Springer resolution

$$\tilde{\mathfrak{g}}_{-\mu} \to \mathfrak{g}$$

where the fibre over $X \in \mathfrak{g}$ is

$$\{gP_{-\mu} \in G/P_{-\mu} : X \in \mathrm{Ad}_g(\mathfrak{p}_{-\mu})\}.$$

Recall from Remark 2.11 that, as μ is minuscule, the evaluation map

$$\mathrm{Gr}_{G,-\mu} \to G/P_{-\mu}$$

is an isomorphism and from the proof of Lemma 2.14 that the affine Springer fibre over $X \in \mathfrak{g}$ corresponds to the partial Springer fibre $\tilde{\mathfrak{g}}_{-\mu}|_X$ above. Therefore, if we consider the pullback

$$\begin{array}{ccc} \tilde{\mathcal{S}}_{-\mu} & \longrightarrow & \tilde{\mathfrak{g}}_{-\mu} \\ {\scriptstyle r_{-\mu}}\downarrow & \lrcorner & \downarrow \\ \mathcal{S} & \hookrightarrow & \mathfrak{g} \end{array},$$

then the image (set theoretic: it is not closed) of the map $\tilde{\mathcal{S}}_{-\mu} \to G/P_{-\mu} \simeq \mathrm{Gr}_{G,-\mu}$ equals precisely $\mathrm{Gr}^{\mathcal{S}}_{G,-\mu}$. Furthermore, the map $r_{-\mu}$ in the previous diagram is finite and we have the following isomorphism diagram [67]:

$$\begin{array}{ccc} \tilde{\mathcal{S}}_{-\mu} & \xrightarrow{\sim} & \mathfrak{t} /\!\!/ W_{L_{-\mu}} \\ {\scriptstyle r_{-\mu}}\downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow{\sim} & \mathfrak{t} /\!\!/ W \end{array}$$

where $W_{L_{-\mu}} \subseteq W$ is the parabolic subgroup corresponding to the Levi subgroup $L_{-\mu} \subseteq P_{-\mu}$ and the top map is constructed by sending $g \in \tilde{\mathfrak{g}}_{-\mu}|_X$ to the image of X via the quotient map

$$\mathfrak{p}' \to \mathfrak{p}' /\!\!/ P' \simeq \mathfrak{t} /\!\!/ W_{L_{-\mu}}$$

where $\mathfrak{p}' = \mathrm{Ad}_g(\mathfrak{p}_{-\mu})$.

Recall from Proposition 2.35 the natural $\mathbb{C}^\times$ action on $\mathrm{Gr}_{G,-\mu} \times \mathcal{S}$ via $\sigma \to \xi_\lambda^{-1}\sigma$ on the first factor and $X \mapsto \lambda\,\mathrm{Ad}_{\xi_\lambda^{-1}}(X)$ on the second). This action restricts to $\tilde{\mathcal{S}}_{-\mu}$, in such a way that $r_{-\mu}$ and the other projection $\tilde{\mathcal{S}}_{-\mu} \to \mathrm{Gr}_{G,-\mu}$ are (tautologically) $\mathbb{C}^\times$-equivariant. In particular, $\mathbb{C}^\times$ acts on the image $\mathrm{Gr}^{\mathcal{S}}_{-\mu} \subseteq \mathrm{Gr}_{G,-\mu}$ and its only fixed point is $z^{-\mu}$, the image of the only fixed point in $\tilde{\mathcal{S}}_{-\mu}$. We can then prove the following.

Proposition 2.37. *Let $\mu \in X_*^+(T)$ be minuscule. Consider the upward flow $W^+_{z^{-\mu}}$ of $z^{-\mu}$ under the $\mathbb{C}^\times$-action on $\mathrm{Gr}^{\mathcal{S}}_G$. Then $\mathrm{Gr}^{\mathcal{S}}_{G,-\mu} \subseteq W^+_{z^{-\mu}}$.*

Proof. Consider the projection $\pi_{-\mu} : \tilde{\mathcal{S}}_{-\mu} \to \mathrm{Gr}_{G,-\mu}$. Since it is $\mathbb{C}^\times$-equivariant, setting $p \in \tilde{\mathcal{S}}^{\mathbb{C}^\times}_{-\mu}$ to be the only fixed point (i.e. p is the only point with $r_{-\mu}(p) = e$) we have

$$\mathrm{Gr}^{\mathcal{S}}_{G,-\mu} = \pi_{-\mu}(W_p^+) \subseteq W^+_{\pi_{-\mu}(p)} = W^+_{z^{-\mu}},$$

giving the inclusion. □

Having understood the *local* picture in the affine Grassmannian, we can now apply it in the moduli space of Higgs bundles. We define the analogue of $\mathrm{Gr}^{\mathcal{S}}_{G,-\mu}$, which is

$$\mathcal{H}_\mu := \left\{(\sigma, \mathcal{E}_a) \in \mathrm{Gr}_{G,-\mu} \times W_0^+ : \sigma \in \mathrm{Gr}_{G,-\mu}^{\varphi_1^a(0)}\right\} \subseteq \mathrm{Gr}_{G,-\mu} \times W_0^+,$$

where the trivialisation of E^0 is the one fixed as before. Combining Proposition 2.37 with Corollary 2.36 and using the fact that the Hecke transformation of $\mathcal{E}^0$ by $z^{-\mu}$ is $\mathcal{E}^\mu$, we get a map

$$\mathcal{H}|_{\mathcal{H}_\mu} : \mathcal{H}_\mu \hookrightarrow W_\mu^+$$

defined by sending $(\sigma, \mathcal{E}_a)$ to the Hecke transformation of $\mathcal{E}^a$ by σ. This map is injective: for two pairs $(\sigma, \mathcal{E}_a)$ and $(\sigma', \mathcal{E}'_a)$, if $a \neq a'$ the two Hecke transformations are different since they lie in different Hitchin fibres and, if $a = a'$ but $\sigma \neq \sigma'$, then the resulting Hecke transformations have non-isomorphic bundles. It is also $\mathbb{C}^\times$-equivariant with respect to the natural action on $\mathcal{H}_\mu$ given by $\mathbb{C}^\times$ acting diagonally on $\mathrm{Gr}_{G,-\mu}$ and W_0^+ (see Proposition 2.35). Finally, $\dim \mathcal{H}_\mu = \dim W_0^+$ (the projection to W_0^+ is finite) and $\dim W_\mu^+ = \dim W_0^+$ since both are Lagrangian.

Proposition 2.38. *The map $\mathcal{H}|_{\mathcal{H}_\mu} : \mathcal{H}_\mu \to W_\mu^+$ given by Hecke transformation is an isomorphism.*

Proof. Since $\mathcal{H}|_{\mathcal{H}_\mu}$ is injective and W_μ^+ is affine, Zariski's main theorem [124, Lemma 37.43.3] applies and there is an open immersion $j : \mathcal{H}_\mu \hookrightarrow Z$ and a finite map $\pi : Z \to W_\mu^+$ such that $\mathcal{H}|_{\mathcal{H}_\mu} = \pi \circ j$. Now, since W_μ^+ is irreducible (it is an affine space) and $\dim Z = \dim \mathcal{H}_\mu = \dim W_\mu^+$, it has dense image. Thus, injectivity of $\mathcal{H}|_{\mathcal{H}_\mu}$ ensures that π is birational. Any birational finite map to a normal variety is an isomorphism, so π is. Therefore, we just need to show that j is an isomorphism. Let $U \subseteq W_\mu^+$ denote the image of j.

Now, $U \subseteq W_\mu^+$ is an open subvariety containing the only fixed point $\mathcal{E}^\mu$, and it is also $\mathbb{C}^\times$-invariant since $\mathcal{H}|_{\mathcal{H}_\mu}$ is $\mathbb{C}^\times$-equivariant. Therefore, it must be the whole of W_μ^+: indeed, if the closed $W_\mu^+ \smallsetminus U$ were nonempty, it would contain the whole $\mathbb{C}^\times$-orbit of any of its elements. But then it would also contain its limit point, contradicting the fact that $\mathcal{E}^\mu \in U$. □

Therefore, the restriction h_μ that we wish to study is isomorphic to the projection $\mathcal{H}_\mu \to W_0^+$. This is already easy to understand, as it comes from the local picture $\tilde{\mathcal{S}}_{-\mu} \to \mathcal{S}$ via evaluation at c. In other words, we have

$$\begin{array}{ccccccc}
W_\mu^+ & \xrightarrow{\sim} & \mathcal{H}_\mu & \xrightarrow{\pi_{\mathrm{Gr}_{G,-\mu}}} & \tilde{\mathcal{S}}_{-\mu} & \xrightarrow{\sim} & \mathfrak{t} /\!\!/ W_{L_{-\mu}} \\
\downarrow{\scriptstyle h_\mu} & & \downarrow{\scriptstyle \pi_{W_0^+}} & \lrcorner & \downarrow{\scriptstyle r_{-\mu}} & & \downarrow \\
\mathcal{A}(G) & \xrightarrow{\sim} & W_0^+ & \xrightarrow{\mathrm{ev}_c} & \mathcal{S} & \xrightarrow{\sim} & \mathfrak{t} /\!\!/ W
\end{array}$$

Completing the desired description. It is worth noting that the projection $\mathfrak{t} /\!\!/ W_{L_{-\mu}} \to \mathfrak{t} /\!\!/ W$ has a geometric interpretation which was studied (in greater generality) in [71]. As a result of Chern–Weil theory, we have

$$\mathfrak{t} /\!\!/ W_{L_{-\mu}} \simeq \mathrm{Spec}\, H^*_{L_{-\mu}}(\mathrm{pt})$$

the $L_{-\mu}$-equivariant cohomology of a point (with complex coefficients, we drop it from the notation since it is our default choice of ring). Using standard arguments in equivariant cohomology, we get

$$H^*_{L_{-\mu}}(\mathrm{pt}) \simeq H^*(BL_{-\mu}) \simeq H^*(BP_{-\mu}) \simeq H^*(EG/P_{-\mu}) \simeq H^*_G(G/P_{-\mu}).$$

Of course, the same thing works for $\mu = 0$ (i.e. $\mathfrak{t} /\!\!/ W \simeq \mathrm{Spec}\, H^*_G(\mathrm{pt})$) so we get the description in terms of the equivariant cohomology of a generalised flag variety as shown by Hausel [67] for $G = \mathrm{GL}_n(\mathbb{C})$:

$$\begin{array}{ccc}
W_\mu^+ & \longrightarrow & \mathrm{Spec}\, H^*_G(G/P_{-\mu}) \\
\downarrow{\scriptstyle h_\mu} & \lrcorner & \downarrow \\
\mathcal{A}(G) & \longrightarrow & \mathrm{Spec}\, H^*_G(\mathrm{pt})
\end{array}.$$

Remark 2.39 *Most of the previous argument carries through whenever μ is not minuscule, by considering the Schubert variety $\mathrm{Gr}_{G,\leqslant\mu} = \overline{\mathrm{Gr}_{G,-\mu}}$ instead. However, the key technical difficulty is that $\overline{\mathrm{Gr}_{G,-\mu}}^{\mathcal{S}}$ is no longer contained in the upward flow $W^+_{z^{-\mu}}$ since now there are many possible limit points. One approach, proposed by Hausel, is to replace the latter upward flow by its closure (in an unusual sense):*

$$\overline{\overline{W^+_{z^{-\mu}}}} = \bigsqcup_{-\lambda\leqslant-\mu} W^+_{z^{-\lambda}}.$$

Then, we would get a result about the Hitchin map restricted to the Lagrangian closure $\overline{\overline{W^+_\mu}}$, meaning the union of all the upward flows W^+_λ with $\lambda \leqslant \mu$. We shall not pursue this further at present.

Remark 2.40 *The morphism of spectra of equivariant cohomologies appearing above can be reinterpreted from representation theory via Hausel's medium algebras [69]. In [69, Theorem 3.1] it is shown that $H^*_G(\overline{\mathrm{Gr}_{G,-\mu}}) \simeq \mathcal{M}^{-\mu}(\mathfrak{g}^\vee)$, where the latter is the medium algebra for $\mathfrak{g}^\vee$ and its weight $-\mu$. This is an algebra over the invariant polynomial ring $\mathbb{C}[\mathfrak{g}^\vee]^{G^\vee}$ and, by the above result, it also models the restricted Hitchin map h_μ in the minuscule upward flows. In our current setting of minuscule μ, moreover, this medium algebra is isomorphic to the more classical family algebra of Kirillov [84] associated to $\mathfrak{g}^\vee$ and $-\mu$. These two isomorphic descriptions of the Hitchin map should reflect the mirror symmetry between $\mathcal{M}(G)$ and $\mathcal{M}(G^\vee)$, although it is not exactly clear how in this generality.*

Chapter 3

Wobbly components for Hermitian-type gradings and the Toledo invariant

3.1 Wobbliness of regular maximal Toledo invariant components via the Cayley correspondence

In this chapter we work with fixed points of the $\mathbb{C}^\times$-action on $\mathcal{M}(G)$ of Hermitian type, as in Definition 1.53. By definition, the corresponding Higgs bundles reduce to Vinberg $\mathbb{C}^\times$-pairs $(G_0, \mathfrak{g}_1)$ for a grading of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with grading element $\zeta \in \mathfrak{t} \subseteq \mathfrak{g}_0$. This situation is different from that of Chapter 2 (except for the rank one case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, where the only Hermitian grading is precisely the Borel grading). In particular, fixed points of Hermitian type do not have regular Higgs fields.

The goal of this section is to exploit the Cayley correspondence of Theorem 1.63 to show that points in $\mathcal{M}^{\max}(G_0, \mathfrak{g}_1)$ are always wobbly, whenever the naturally associated real form $G_\mathbb{R} \subseteq G$ of Hermitian type, defined by having complexified Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \oplus \mathfrak{g}_1),$$

is not a split real form. This happens for most Hermitian gradings, except for those corresponding to Hermitian real forms $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SU}(1,1)$, $\mathrm{SO}(2,1)$ and $\mathrm{SO}(2,2)$ (when $\mathfrak{g}$ is simple, also note that some of those are isomorphic). Interestingly, for these split cases, the uniformising Higgs bundles have maximal Toledo invariant, so they are contained in $\mathcal{M}^{\max}(G_\mathbb{R})$. For the rank 2 examples they are also contained in $\mathcal{M}^{\max}(G_0, \mathfrak{g}_1)$, so such a theorem cannot be expected since these are very stable, but it would be interesting to study the higher rank cases with $G_\mathbb{R} = \mathrm{Sp}(2n, \mathbb{R})$ since a priori there is no reason for them to be wobbly or very stable.

We assume that $(G_0, \mathfrak{g}_1)$ is JM-regular (see Definition 1.61) since otherwise $\mathcal{M}^{\max}(G_0, \mathfrak{g}_1)$ does not contain stable Higgs bundles [14, Proposition 6.15], therefore it does not contain very stable Higgs bundles by definition. Let $e \in \Omega \subseteq \mathfrak{g}_1$ be an element of the open orbit and complete it $(e, 2\zeta, f)$ to a $\mathfrak{sl}_2$-triple with $f \in \mathfrak{g}_{-1}$. We denote by G_0^e the centraliser of e in G_0; it is reductive. The Cayley correspondence of Theorem 1.63 establishes an isomorphism

$$\mathcal{M}^{max}(G_0, \mathfrak{g}_1) \simeq \mathcal{N}(G_0^e),$$

where $\mathcal{N}(G_0^e)$ is the moduli space of isomorphism classes of polystable principal G_0^e-bundles over C.

Recall from Section 1.4 that the isomorphism is given by fixing a T-bundle E_T satisfying $E_T(\langle e \rangle)) \simeq K^{-1}$ (this is the uniformising bundle of [14, Example 4.4]) and mapping a G_0^e-principal bundle E' to $(E_T \otimes E')(G_0)$ with the *constant* Higgs field e. Conversely, given (E, φ) with maximal Toledo invariant, the Higgs field takes values in Ω everywhere, giving a reduction of structure group of $E_T^{-1} \otimes E$ to G_0^e.

This isomorphism can be upgraded to the level of $G_{\mathbb{R}}$-Higgs bundles (in fact, as explained in Section 1.4, the latter predates [16, Theorem 5.2] the former) by considering the subspace $V \subseteq \mathfrak{g}_0$ defined by

$$V := \mathrm{ad}(e)(\mathfrak{g}_{-1}).$$

It turns out that $V \simeq \mathfrak{g}_{-1}$ via $\mathrm{ad}(e)$, and that $\mathfrak{g}_0$ splits

$$\mathfrak{g}_0 = \mathfrak{g}_0^e \oplus V$$

and this splitting is a complexified Cartan decomposition for a real form $G_0^{\mathbb{R}} \subseteq G_0$ known as the **Cayley partner**.

Theorem 3.1 (Cayley correspondence for $G_{\mathbb{R}}$-Higgs bundles in the JM-regular (*tube type*) case [16, Theorem 5.2]). *There is an isomorphism*

$$\mathcal{M}(G_{\mathbb{R}}) := \mathcal{M}^{max}(G_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_1) \simeq \mathcal{M}_{K_C^2}(G_0^e, V) = \mathcal{M}_{K_C^2}(G_0^{\mathbb{R}}),$$

which restricts to the isomorphism of Theorem 1.63.

In the theorem statement, the subscript K_C^2 indicates that the corresponding moduli spaces parametrise K_C^2-twisted Higgs pairs, meaning that the Higgs field is twisted by K_C^2 instead of K_C. The Cayley correspondence map is defined similarly, but now the extra data $\varphi^- \in H^0(E(\mathfrak{g}_{-1}) \otimes K_C)$ is sent to $\psi := \mathrm{ad}(\varphi)(\varphi^-) \in H^0(E'(V) \otimes K_C^2)$. Conversely, from (E', ψ) one recovers φ^- as the unique section with $\mathrm{ad}(e)(\varphi^-) = \psi$, using the fact that $\mathfrak{g}_{-1}$ and V are G_0^e-equivariantly isomorphic.

This correspondence satisfies two properties which are crucial for our purposes.

Proposition 3.2. *1. The Cayley correspondence is $\mathbb{C}^\times$-equivariant with respect to the $\mathbb{C}^\times$-action on $\mathcal{M}_{K_C^2}(G_0^e, V)$ given by $(E', \psi) \mapsto (E', \lambda^2\psi)$.*

2. The Cayley correspondence maps Hitchin fibres to Hitchin fibres. In doing so, it induces a linear identification between both Hitchin bases.

The first statement is evident from the definition of the Cayley map, since $\mathrm{ad}(\lambda\varphi)(\lambda\varphi^-) = \lambda^2 \mathrm{ad}(\varphi)(\varphi^-)$. The second statement is [25, Proposition 7.16].

Corollary 3.3. *If $(E, \varphi) \in \mathcal{M}^{max}(G_0, \mathfrak{g}_1)$ maps to a (V, K_C^2)-wobbly bundle $E' \in \mathcal{N}(G_0^e)$, meaning that there is a nonzero nilpotent section $\psi \in H^0(E(V) \otimes K_C^2)$, then (E, φ) is wobbly.*

Proof. The $G_0^{\mathbb{R}}$-Higgs bundle (E', ψ) is nilpotent in the upward flow of $(E', 0)$. By Proposition 3.2, the corresponding $(E, \varphi + \varphi^-)$ is nilpotent (by property (2)) in the upward flow of (E, φ) (by property (1)). □

We can therefore study wobbliness of (E, φ) by analysing the (V, K_C^2)-wobbliness of E'.

Proposition 3.4. *Suppose that $G_0^{\mathbb{R}}$ is not split. Let $E' \in \mathcal{N}^s(G_0^e)$ be stable. Then, it is (V, K_C^2)-wobbly, i.e. there is a nonzero nilpotent section $\psi \in H^0(E'(V) \otimes K_C^2)$.*

Proof. We consider the algebra of invariant polynomials $\mathbb{C}[V]^{G_0^e} = \mathbb{C}[p_1, \ldots, p_r]$ which is free and finitely generated by r elements. These r generators give a map

$$H^0(E'(V) \otimes K_C^2) \to \mathcal{A}(G_0^{\mathbb{R}}) = \bigoplus_{i=1}^{r} H^0(K_C^{2\deg(p_i)}).$$

Since this is an algebraic map between two vector spaces and 0 is in the image, it suffices to check that the dimension of the source is strictly larger than the dimension of the target.

By Riemann–Roch, using that $\deg(E'(V)) = 0$ (because the Killing form of $\mathfrak{g}_0$ induces an isomorphism between $E'(V)$ and its dual), and that $H^1(E'(V) \otimes K_C^2) \simeq H^0(E'(V)^* \otimes K_C^{-1})^* = 0$ because E' is stable and $\deg(E'(V)^* \otimes K_C^{-1}) < 0$, we get

$$\dim H^0(E'(V) \otimes K_C^2) = \dim(V)(3g-3).$$

Similarly, we get

$$\dim H^0(K_C^{2\deg(p_i)}) = (4g-4)\deg(p_i) - (g-1)$$

$$\dim \mathcal{A}(G_0^{\mathbb{R}}) = (4g-4)\sum_{i=1}^{r} \deg(p_i) - r(g-1).$$

Therefore, we need the difference

$$(4g-4)\left(\dim V - \sum_{i=1}^{r} \deg(p_i)\right) + (r - \dim(V))(g-1)$$

to be positive, i.e. we need

$$4 > \frac{\dim(V) - r}{\dim(V) - \sum_{i=1}^{r} \deg(p_i)} =: \eta$$

(the denominator is nonzero precisely if and only if $G_0^{\mathbb{R}}$ is nonsplit).

It suffices to check this inequality of every simple Lie type using [16, Table 4]. Indeed, if $G_0^{\mathbb{R}}$ had several simple components, we could apply the argument to one of the non-split ones to construct the nilpotent Higgs field. We see from Table 3.1 that the result is always lower than four in the non-split cases.

Table 3.1 Value of η for Hermitian forms of tube type.

$G_{\mathbb{R}}$	$G_0^{\mathbb{R}}$	$\dim(V)$	η
$\mathrm{SU}(n,n)$	$\{A \in \mathrm{GL}_n(\mathbb{C}) : \det(A)^2 \in \mathbb{R}^+\}$	n^2	2
$\mathrm{SO}^*(4n)$	$U^*(2n)$	$2n^2 - n$	$\frac{4}{3}$
$\mathrm{SO}_0(2n)$	$\mathrm{SO}_0(1,1) \times SO(1,n-1)$	n	$\frac{n-2}{n-3}$
E_7^{-25}	$E_6^{-26} \ltimes \mathbb{R}^*$	27	$\frac{8}{7}$

□

We can now combine all the previous results to conclude the following.

Theorem 3.5. *Let $(E,\varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ be a fixed point of Hermitian type whose associated Hermitian grading is JM-regular and such that $G_{\mathbb{R}} \subseteq G$ is not split. Suppose that $\tau(E,\varphi) = -\tau_{\max} = -(2g-2)\,\mathrm{rk}_T(G_0,\mathfrak{g}_1)$. Then, (E,φ) is wobbly.*

Proof. This is a direct consequence of Corollary 3.3 and Proposition 3.4, by noticing (e.g. [16, Table 4]) that the splitness of $G_{\mathbb{R}} \subseteq G$ is equivalent to that of $G_0^{\mathbb{R}} \subseteq G_0$ and that, whenever (E,φ) is stable (which we may assume since otherwise it is automatically wobbly), the corresponding $E' \in \mathcal{N}(G_0^e)$ is also stable [14, Proof of Theorem 6.2]. □

This strategy can also be exploited to study JM-regular fixed points corresponding to arbitrary gradings of $\mathfrak{g}$, for example by using the more general Cayley correspondence for cyclic Higgs bundles of [50]. We discuss some ideas for this general case in the last chapter of the thesis.

3.2 Wobbliness of non-regular large Toledo invariant components

In this section we consider Hermitian gradings $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which are not JM-regular. We shall find a non-empty region of the form $[-\tau_{\max}, \tau')$ such that every fixed point (E, φ) whose Toledo invariant lies in this region is automatically wobbly.

Our strategy is to start from a $(G_0, \mathfrak{g}_1)$-Higgs pair (E, φ) defining a smooth point of $\mathcal{M}(G)$ fixed by the $\mathbb{C}^\times$-action and construct a nonzero $\beta \in H^0(E(\mathfrak{g}_{-1}) \otimes K_C)$ such that $\varphi + \beta$ is nilpotent. By Proposition 1.71, the Higgs bundle $(E, \varphi + \beta)$ lies in the upward flow $W^+_{(E,\varphi)}$, implying the wobbliness of (E, φ).

Thus the first natural question is the local one, i.e. consider $e \in \Omega_\varphi \subseteq \mathfrak{g}_1$ an element in the open orbit taken generically by φ and find elements $Y \in \mathfrak{g}_{-1}$ such that $e + Y$ is nilpotent. We may use the following general criterion.

Lemma 3.6. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and $\mathfrak{p} \subseteq \mathfrak{g}$ a parabolic subalgebra with nilpotent radical $\mathfrak{n} \subseteq \mathfrak{p}$. Let $X \in \mathfrak{p}$ be a nilpotent element of $\mathfrak{g}$ contained in $\mathfrak{p}$ and $Y \in \mathfrak{n}$. Then $X + Y$ is a nilpotent element of $\mathfrak{g}$.*

Proof. Parabolic subalgebras of a semisimple Lie algebra $\mathfrak{g}$ always induce a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{1-m} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{m-1},$$

for some integer $m \geqslant 1$, such that $\mathfrak{p} = \bigoplus_{j \geqslant 0} \mathfrak{g}_j$ and $\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}_j$. For example, this grading can be obtained via the description of parabolic subalgebras (up to conjugation) from subsets of the simple roots explained in Section 1.1.

Write $A := \operatorname{ad} X$ and $B := \operatorname{ad} Y$. Since $Y \in \mathfrak{n}$, the previous description implies that for any string

$$S := A^{l_1} B^{r_1} A^{l_2} \dots B^{r_k}$$

obtained by repeatedly composing A and B, if the number $r := \sum r_j$ of occurrences of B satisfies $r \geqslant 2m - 1$, then $S = 0$.

Thus, when expanding $(A + B)^M$ we may assume that the strings of length M that we obtain have less than $2m - 1$ instances of B. Hence, given any positive integer K there exists M large enough so that every string contains an occurrence of A^K. In particular, if K is the nilpotency order of X, we get that $(A + B)^M = 0$. □

We can find elements $Y \in \mathfrak{g}_{-1}$ to which this lemma can be applied by considering a further decomposition of $\mathfrak{g}$ which exists for non-JM-regular gradings. Indeed, taking $e \in \Omega \subseteq \mathfrak{g}_1$ an element of the open orbit, we can complete it to a $\mathfrak{sl}_2(\mathbb{C})$-triple (e, h, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$. Since $[\zeta, h] = 0$, we can decompose each piece of the gradings in $\operatorname{ad}(h)$-eigenvalues:

$$\mathfrak{g}_j := \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_j^k.$$

This decomposition endows $\mathfrak{g}$ with the structure of a **$\mathbb{Z}^2$-grading (or bigrading)**.

Let $\hat{\mathfrak{g}}_j := \mathfrak{g}_j^{2j}$ and $\mathfrak{g}'_j := \bigoplus_{k \neq 2j} \mathfrak{g}_j^k$, so that $\mathfrak{g}_j = \hat{\mathfrak{g}}_j \oplus \mathfrak{g}'_j$. Let

$$\hat{\mathfrak{g}} := \hat{\mathfrak{g}}_{-1} \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_1.$$

It is clear that $\hat{\mathfrak{g}} \subseteq \mathfrak{g}$ is a Lie subalgebra with a $\mathbb{Z}$-grading of Hermitian type, and the corresponding subpair $(\hat{G}_0, \hat{\mathfrak{g}}_1) \subseteq (G_0, \mathfrak{g}_1)$ is the **maximal JM-regular prehomogeneous vector subspace** of [14, Definition 2.16]. It is clearly JM-regular as we constructed it by selecting the pieces where $\operatorname{ad}(\zeta)$ agrees with $\operatorname{ad}(h)/2$. We are interested in the non JM-regular part $\mathfrak{g}'_{-1}$.

Lemma 3.7. *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Hermitian $\mathbb{Z}$-grading and $e \in \mathfrak{g}_1$. Let $Y \in \mathfrak{g}'_{-1}$. Then $e + Y$ is nilpotent.*

Proof. Complete e to a $\mathfrak{sl}_2(\mathbb{C})$-triple (e, h, f) with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$. Consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ given by the eigenvalues of $\eta := h - 2\zeta$. Since $[\eta, e] = 2 - 2 = 0$, e belongs to the parabolic $\mathfrak{p}$ of the grading by η. Moreover, since η acts with positive eigenvalues on $\mathfrak{g}'_1$ (as $\mathrm{ad}(e)^3 = 0$ implies that the eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{g}$ are greater or equal than -2), we have that Y belongs to the nilpotent radical of $\mathfrak{p}$. Thus, Lemma 3.6 applies. □

The last step is to solve the global problem of finding sections of $E(\mathfrak{g}'_{-1}) \otimes K_C$. This can be done whenever the Toledo invariant is large.

Theorem 3.8. *Let (E, φ) be a stable fixed point of Hermitian type with Toledo invariant $\tau(E, \varphi)$. Use an element $e \in \mathfrak{g}_1$ of the G_0-orbit taken generically by φ to construct $\hat{\mathfrak{g}}_j$ and $\mathfrak{g}'_{-1}$. Let $\hat{B}^*$ and $\hat{\beta}$ be the corresponding objects for $\hat{\mathfrak{g}}$ in the definition of its Toledo character. Assume that*

$$\tau(E, \varphi) < 2B^*(\beta, \beta)(g-1)\left(\frac{-\operatorname{rk}_T(\varphi)}{\hat{B}^*(\hat{\beta}, \hat{\beta})} + \dim \mathfrak{g}'_{-1}\right).$$

Then (E, φ) is wobbly.

Proof. We start recalling a construction introduced in [14, Proof of Theorem 5.3]. Let $e \in \mathfrak{g}_1$ be the element in the G_0-orbit $\Omega_\varphi \subseteq \mathfrak{g}_1$ taken generically by φ. Using e, construct $\hat{\mathfrak{g}}_j$ and $\mathfrak{g}'_{-1}$ from the $\mathfrak{sl}_2$-triple (h, e, f) as before. Since φ takes values generically in Ω_φ, if we let $P_{0,e} \subseteq G_0$ be the parabolic subgroup whose Lie algebra is the associated to the grading by $\eta := h - 2\zeta$, then φ defines a section $\sigma^* \in H^0(C^*, E(G_0/P_{0,e}))$ on C^* the complement of the divisor where φ exits the orbit Ω_φ.

Since $G_0/P_{0,e}$ is projective this extends to $\sigma \in H^0(C, E(G_0/P_{0,e}))$ a reduction of structure group of E to $P_{0,e}$. Furthermore, projecting to the Levi factor of $P_{0,e}$, which is $\hat{G}_0$, gives a principal $\hat{G}_0$-bundle $E_\sigma(\hat{G}_0)$ with a well defined section $\varphi \in H^0(C, E_\sigma(\hat{G}_0)(\hat{\mathfrak{g}}_1) \otimes K_C)$. Over C^*, the pair $(E_\sigma(\hat{G}_0), \varphi)$ is a honest reduction of (E, φ) to $(\hat{G}_0, \hat{\mathfrak{g}}_1)$.

Having recalled this construction, we will show that there exists a nonzero section β of $E_\sigma(\mathfrak{g}'_{-1} \otimes K_C)$ (notice that $P_{0,e}$ acts on $\mathfrak{g}'_{-1}$ by restricting the adjoint representation) and thus, by Lemma 3.7, we will be able to conclude that $\varphi + \beta$ is a nilpotent Higgs field on E. Moreover, Proposition 1.71 shows that this Higgs field defines a point in the upward flow of (E, φ) different from it, and it is stable by openness and $\mathbb{C}^\times$-invariance of stability as (E, φ) is.

The existence of a nonzero section of $E_\sigma(\mathfrak{g}'_{-1}) \otimes K_C$ is a Riemann–Roch argument. We compute

$$\deg(E_\sigma(\mathfrak{g}'_{-1}) \otimes K_C) = \deg E_\sigma(\mathfrak{g}'_{-1}) + 2(g-1)\dim \mathfrak{g}'_{-1}.$$

Now,

$$\deg E_\sigma(\mathfrak{g}'_{-1}) = \deg E_\sigma(\mathfrak{g}_{-1}) - \deg E_\sigma(\mathfrak{g}_{-1}/\mathfrak{g}'_{-1}) = -\frac{\tau(E, \varphi)}{2B^*(\beta, \beta)} - \deg E_\sigma(\hat{G}_0)(\hat{\mathfrak{g}}_{-1}).$$

Using the fact that $\deg E_\sigma(\hat{G}_0)(\hat{\mathfrak{g}}_{-1}) = -\deg E_\sigma(\hat{G}_0)(\hat{\mathfrak{g}}_1)$ (this comes from the pairing given by the Killing form, which is non-degenerate on $\hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_{-1}$ since it pairs eigenspaces for eigenvalues with opposite sign, both for ζ and for h), and bounding $\deg E_\sigma(\hat{G}_0)(\hat{\mathfrak{g}}_1)$ as in [14, Proof of Theorem 5.3] (by means of certain *relative invariant* for the JM-regular prehomogeneous vector space $(\hat{G}_0, \hat{\mathfrak{g}}_1)$), we get

$$\deg E_\sigma(\mathfrak{g}'_{-1}) \geqslant -\frac{\tau(E,\varphi)}{2B^*(\beta,\beta)} - (g-1)\frac{\mathrm{rk}_T(\varphi)}{\hat{B}^*(\hat{\beta},\hat{\beta})},$$

where $\hat{B}$ and $\hat{\beta}$ are the corresponding objects in $\hat{\mathfrak{g}}$.

Now, Riemann–Roch gives

$$\chi(E_\sigma(\mathfrak{g}'_{-1}) \otimes K_C) \geqslant -\frac{\tau(E,\varphi)}{2B^*(\beta,\beta)} - (g-1)\frac{\mathrm{rk}_T(\varphi)}{\hat{B}^*(\hat{\beta},\hat{\beta})} + 2(g-1)\dim \mathfrak{g}'_{-1} - (g-1)\dim \mathfrak{g}'_{-1} > 0,$$

where the positivity is just the constraint in the proposition.

□

Remark 3.9 *As a sanity check, the previous proposition is of no use in the JM-regular case. Indeed, the resulting condition is $\tau < -(2g-2)\,\mathrm{rk}_T(\varphi)$, which does not happen by Theorem 1.60. This is expected as $\mathfrak{g}'_{-1} = 0$.*

In fact, since $\hat{B}^*(\hat{\gamma},\hat{\gamma}) \geqslant B^*(\gamma,\gamma)$ (as can be seen, for example, by writing $B, \hat{B}, \gamma$ and $\hat{\gamma}$ in terms of root data, or relating these numbers with the dual Coxeter number as in [16, Definition 2.4]) and $\dim \mathfrak{g}'_{-1} \geqslant 0$, both with equality if and only if the pair is JM-regular, we deduce that in the non-regular case there always exists $\tau' > -\tau_{max}$ such that the components with Toledo invariant in the interval $[-\tau_{max}, \tau')$ are wobbly.

Remark 3.10 *In type A, corresponding to the real form $G_{\mathbb{R}} = SU(p,q)$ as in Example 1.55, we get the following condition for wobbliness:*

$$\tau < -(2g-2)\frac{3(\min\{p,q\})^2 - pq}{p+q}.$$

This matches one of the conditions for wobbliness [109, Theorem 6.1].

Remark 3.11 *By the Arakelov–Milnor inequality of Theorem 1.60, we have already proven wobbliness of all components in those cases where*

$$\dim \mathfrak{g}'_{-1} > \frac{\mathrm{rk}_T(G_0,\mathfrak{g}_1)}{\hat{B}^*(\hat{\gamma},\hat{\gamma})}.$$

For example, for simple Lie algebra types, this happens precisely with fixed points for $G = \mathrm{SL}_n(\mathbb{C})$ of type (p,q) where

$$q > 3p.$$

This was already remarked in [70, Remark 5.13].

Note that, in the previous remarks, we have taken $e \in \Omega$ for the computations. Indeed, generically on a given component of the fixed point locus the Higgs field takes values in Ω (one strategy to see this, which we shall not develop here, is to vary the stability parameter that appears in the study of $(G_0,\mathfrak{g}_1)$-pairs, which results in birationally equivalent moduli spaces, and then realise that large values of the parameter force this condition), and being very stable is an open condition. This justifies that if the bound of Theorem 3.8 for $e \in \Omega$ is satisfied, the component does not contain very stable Higgs bundles.

Many of the considerations in this section can be exploited in order to study fixed points corresponding to arbitrary gradings of $\mathfrak{g}$. For example, the core Lemma 3.6 is stated in general terms and does not assume parabolics of Hermitian type. We discuss the present status of this study and remaining steps in the last chapter of this document, concerning future work.

Chapter 4

Very stable parabolic G-Higgs bundles and affine flag varieties

4.1 Affine flag varieties and Hecke transformations of parabolic G-Higgs bundles

In order to understand the space of Hecke transformations in the parabolic setting, we first recall the relevant notions about affine flag varieties. Our main references are [106, 110, 134].

We follow the same notation as in Section 2.1, for example $\mathbb{D} := \operatorname{Spec} \mathbb{C}[[z]]$ denotes the formal disk and $\mathbb{D}^* := \operatorname{Spec} \mathbb{C}((z))$ the punctured disk. For parabolic bundles and parabolic Higgs bundles we follow the same notation and conventions as in Section 1.6. In particular, we still assume that $\alpha \in \mathfrak{t}_{\mathbb{R}}$ is in the interior of the standard Weyl alcove.

Definition 4.1 *The **affine flag variety** Fl_G is defined by its functor of points, assigning to a $\mathbb{C}$-algebra R the space of isomorphism classes of triples (E, Q, Ψ) where (E, Q) is a parabolic G-bundle with weight α over $(\mathbb{D}_R, 0_R)$ and $\Psi : E|_{\mathbb{D}_R^*} \to \mathbb{D}_R^* \times G$ is a trivialisation over $\mathbb{D}_R^*$.*

In this setting, the parabolic structure Q corresponds to a reduction of structure group of E to B over 0_R. As with Gr_G, the affine flag variety Fl_G also has the structure of an ind-projective ind-scheme, and there is a forgetful map $(E, Q, \Psi) \mapsto (E, \Psi)$ defining a projection $\mathrm{Fl}_G \to \mathrm{Gr}_G$.

In order to describe Fl_G in terms of the loop group $G((z))$, it suffices to replace $G[[z]]$ by the **standard Iwahori subgroup** $\mathcal{I} \subseteq G[[z]]$, which is the preimage of $B \subseteq G$ via the evaluation map

$$G[[z]] \to G.$$

We then have an identification of the $\mathbb{C}$-points of Fl_G with the quotient $G((z))/\mathcal{I}$. This is completely analogous to the identification of Gr_G with $G((z))/G[[z]]$ in Section 2.1, but now the trivialisation $\varepsilon : \mathbb{D} \times G \xrightarrow{\sim} E$ is required to send $0 \times B$ to $Q \subseteq E|_0/B$. Thus, a change in trivialisation results in a change by an element of $\operatorname{Aut}(\mathbb{D} \times G, 0 \times B) \simeq \mathcal{I}$ instead.

As in the case of Gr_G, we will typically denote the space of $\mathbb{C}$-points by Fl_G as well, while most of the results stated here for the affine flag variety hold schematically.

Remark 4.2 *For more general parabolic weights $\alpha \in \mathfrak{t}_{\mathbb{R}}$, everything works identically by replacing parabolic bundles with parahoric torsors [5] as indicated in Remark 1.75. The subgroups of $G((z))$ that appear as automorphisms of the trivial parahoric torsor over $\mathbb{D}$ are the **parahoric subgroups** $G_\alpha \subseteq G((z))$, defined by restricting their valuation along each direction of the root space decomposition: they have valuation at least 0 along T, and at least $\lceil -\beta(\alpha) \rceil$ along the root subgroup for $\beta \in \Delta$. In particular, for $\alpha = 0$ we have $G_\alpha = G[[z]]$ and for α in the interior of the standard Weyl alcove we have $G_\alpha = \mathcal{I}$, recovering our two main cases of interest.*

Now we recall the Bruhat decomposition for Fl_G. This can be done in the general setting of $\mathrm{Fl}_{G,\alpha} := G((z))/G_\alpha$ (see Remark 4.2) for any $\alpha \in \mathfrak{t}_\mathbb{R}$: the case $\alpha = 0$ recovers the Cartan decomposition of Gr_G from Section 2.1 and any choice of α in the interior of the standard Weyl alcove recovers Fl_G.

Recall that we can regard $X^*(T) \subseteq G((z))$. Moreover, since $T \subseteq G_\alpha$, we may map the Weyl group $W = N(T)/T$ into Fl_α. This gives a way to map

$$\widetilde{W}_G = X_*(T) \rtimes W \to \mathrm{Fl}_\alpha\,.$$

Remark 4.3 *At this point we can clarify the sign convention in Remark 1.101. We want our action of $X_*(T)$ on $\mathfrak{t}$ to satisfy $G_{\mu(\alpha)} = z^\mu G_\alpha z^{-\mu}$. Since $G_{\alpha=0} = G[[z]]$, z^μ sends 0 to the point stabilised by*

$$z^\mu G[[z]] z^{-\mu}.$$

Along the direction of the root $\beta \in \Delta$, this group has elements with valuation at least $\beta(\mu)$. Now, the stabiliser of a point $\alpha \in \mathfrak{t}$ is by definition the parahoric subgroup G_α, whose elements along that direction have valuation at least $-\beta(\alpha)$. We therefore conclude that z^μ maps 0 to $-\mu$. Another common way around this is to change the sign in the definition of z^μ instead of in the action of $\widetilde{W}_G$.

When understood, we will abbreviate $\mathrm{Fl}_\alpha := \mathrm{Fl}_{G,\alpha}$.

Definition 4.4 *Let $w \in \widetilde{W}_G$. We define the **Bruhat cell** of w in Fl_α as*

$$\mathrm{Fl}_{\alpha,w} := G_\alpha \cdot w \subseteq \mathrm{Fl}_\alpha\,.$$

*We define the **Schubert variety** of w in Fl_α as*

$$\mathrm{Fl}_{\alpha,\leqslant w} := \overline{\mathrm{Fl}_{\alpha,w}} \subseteq \mathrm{Fl}_\alpha\,.$$

For what follows, recall the Bruhat order on $\widetilde{W}_G$ from Definition 1.3 and the parabolic quotients in Definition 1.4 and Remark 1.6.

Proposition 4.5 ([81], [110, Proposition 0.1]). *Let $\alpha, \alpha' \in \mathfrak{t}_\mathbb{R}$ be parabolic weights. We have the Cartan–Iwahori–Matsumoto decomposition in Bruhat cells:*

$$\mathrm{Fl}_\alpha := \bigsqcup_{w \in {}_{J_{\alpha'}}\widetilde{W}_G^{J_\alpha}} G_{\alpha'} \cdot w.$$

Furthermore, when $\alpha = \alpha'$, we have the following description of the Schubert varieties:

$$\mathrm{Fl}_{\alpha,\leqslant w} = \overline{\mathrm{Fl}_{\alpha,w}} = \bigsqcup_{w' \leqslant w} \mathrm{Fl}_{\alpha,w'}$$

and their dimension equals the length $l(w)$.

In our main case of interest, when α is in the interior of the standard alcove, we get $J_\alpha = \emptyset$, so $(\widetilde{W}_G)_{J_\alpha} = 1$ and the relevant (left, right or double) quotient is ${}_{J_\alpha}(\widetilde{W}_G)^{J_\alpha} = \widetilde{W}_G$ with its Bruhat order. Recall also from Section 2.1 that, in the case of $\alpha = 0$, the double quotient becomes $X_*^+(T)$ with the usual dominance order.

Affine flag varieties are related to local transformations of parabolic G-bundles, but in the presence of a Higgs field we need to look at affine Springer fibres. We consider the infinitesimal versions

of the loop groups above, namely the loop Lie algebra $\mathfrak{g}((z)) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ and its parahoric subalgebras for $\alpha \in \mathfrak{t}_{\mathbb{R}}$ given by

$$\mathfrak{g}_\alpha = \mathfrak{t}[[z]] \oplus \bigoplus_{\beta \in \Delta} z^{\lceil -\beta(\alpha) \rceil} \mathfrak{g}_\beta[[z]] \subseteq \mathfrak{g}((z)).$$

For $\alpha = 0$ we recover the positive loop Lie algebra

$$\mathfrak{g}[[z]] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[z]]$$

and for α in the interior of the standard Weyl alcove we recover the standard Iwahori Lie algebra $\mathfrak{i} \subseteq \mathfrak{g}((z))$. Note that, when considering Higgs fields over $\mathbb{D}$, they are simply sections in $H^0(\mathbb{D}, E(\mathfrak{g}))$ without further twisting.

Definition 4.6 *Let $X \in \mathfrak{i}$. The moduli space of isomorphism classes of pairs $((E, \varphi, Q), \Psi)$ where (E, φ, Q) is a parabolic G-Higgs bundle over $\mathbb{D}$ of weight α in the interior of the standard Weyl alcove and $\Psi : (E, \varphi, Q)|_{\mathbb{D}^*} \xrightarrow{\sim} (\mathbb{D}^* \times G, X, 0 \times B)$ is a trivialisation over $\mathbb{D}^*$ is called the **affine Springer fibre** of X in Fl_G and denoted Fl_G^X.*

There is an inclusion of Fl_G^X into Fl_G given by forgetting φ, and we have the natural identification:

$$\mathrm{Fl}_G^X := \{\sigma \in \mathrm{Fl}_G : \mathrm{Ad}_{\sigma^{-1}}(X) \in \mathfrak{i}\} \subseteq \mathrm{Fl}_G\,.$$

As for Gr_G, the latter follows by taking $((E, Q), \Psi)$ and imposing that $\Psi^*(X)$ extends to $0 \in \mathbb{D}$ and is compatible with the parabolic structure. It is clear from these descriptions how affine Springer fibres can be generalised to other $\alpha \in \mathfrak{t}_{\mathbb{R}}$, and the modular interpretation corresponds to that of *logahoric Higgs torsors* [91] over $\mathbb{D}$ with trivialisation over $\mathbb{D}^*$.

For our purposes with fixed points of Borel type, it is interesting as a first step to understand which elements in $\widetilde{W}_G$ lie in the affine Springer fibre in Fl_G for the regular nilpotent $e = \sum_{j=1}^r X_{\beta_j}$. It suffices to understand the *larger* case of $\widetilde{W}_{G_{ad}} = \widetilde{W}$.

Lemma 4.7. *An element $w = \mu \cdot u \in \widetilde{W} = P_{\Delta^\vee} \rtimes W$ belongs to $\mathrm{Fl}^e_{G_{\mathrm{ad}}}$ if and only if μ is antidominant and $u \in {}^{J_\mu}W$. Equivalently, if and only if $w \in {}^{S_W}\widetilde{W}$.*

Proof. Write $\mu = \sum_{i=1}^r n_i \omega_i^\vee$ in terms of the fundamental coweight basis. Then

$$\mathrm{Ad}_{u^{-1}z^{-\mu}}(e) = \mathrm{Ad}_{u^{-1}}(\sum_{i=1}^r z^{-n_i} X_{\beta_i}) = \sum_{i=1}^r z^{-n_i} X_{u^{-1}(\beta_i)}.$$

For this to be in the standard Iwahori subalgebra in particular it has to be in $\mathfrak{g}[[z]]$, so $n_i \leqslant 0$ for all i and μ is antidominant. Moreover, we need that $u^{-1}(\beta_i)$ is a positive root whenever $n_i = 0$. This can be restated as $l(u^{-1}s_{\beta_i}) > l(u^{-1})$ whenever $n_i = 0$. Inverting, this is the same as $l(s_{\beta_i}u) > l(u)$ for all β_i with $\beta_i(\mu) = 0$, as desired. □

Remark 4.8 *The general philosophy, at least for the adjoint group G_{ad}, is that every fixed point of Borel type can be obtained by Hecke transformations of the canonical uniformising Higgs bundle (i.e. everywhere regular nilpotent Higgs field) in Example 1.114. The local picture around a parabolic puncture, as we will detail in the next subsection, is therefore given by $\mathrm{Fl}^e_{G_{ad}}$ from previous Lemma, which is therefore interesting to compare with Remark 1.102.*

Now we describe Hecke transformations for parabolic G-Higgs bundles at parabolic points $c \in C$. Due to the local nature of the operation, at non-parabolic points the analysis is the exactly the same as in Section 2.1.

We start from a parabolic G-Higgs bundle (E, φ, Q). Following the same notation as in Section 2.1, we start by understanding the local data. Fortunately, parabolic structures are local in nature, so if $c_i \neq c$ we can identify $E|_{c_i}/B \simeq E_0|_{c_i}/B$ and if $c_i = c$ we can identify $E|_{c_i}/B \simeq E_1|_{c_i}/B$. The local data is then

$$(E_0, E_1, \Phi, \varphi_0, \varphi_1, Q).$$

The only main difference when considering Hecke transformations for parabolic G-bundles and G-Higgs bundles is that bundle isomorphisms must be compatible with the parabolic structures.

Definition 4.9 *Let (E, φ, Q) be a parabolic G-Higgs over C. A **Hecke transformation** of (E, φ, Q) at $c \in C$ is another parabolic G-Higgs bundle (E', φ', Q') over C together with a parabolic isomorphism*

$$\Psi : E'|_{C_0} \to E|_{C_0}$$

*such that $\varphi_0 \circ \Psi = \varphi'_0$. The isomorphism being parabolic means that $\Psi|_{c_i}(Q'_i) = Q_i$ for every $c_i \in D \smallsetminus \{c\}$. Two Hecke transformations (E', φ', Ψ) and (E'', φ'', Ψ') are **equivalent** if and only if there is a parabolic isomorphism $E' \xrightarrow{f} E''$ such that $\Psi' \circ f|_{C_0} = \Psi$ and $\varphi'' \circ f = \varphi'$.*

The study of Hecke transformations mimics the case without parabolic structures, so we only mention the differences. If $c \notin D$, everything is identical. If $c \in D$ (say, without losing generality, that $c = c_1$) we fix a trivialisation

$$t_1 : E_1 \to C_1 \times G$$

that sends the parabolic structure to the trivial B-orbit, i.e. such that $t_1|_{c_1}(Q_1) = B \in (C_1 \times G)|_{c_1}/B = G/B$. Then, since B is self-normalising, we have that $\mathrm{Aut}(C_1 \times G, B) \simeq \mathcal{I}$ and the compatibility of the Higgs field with the parabolic structure at c_1 translates to $\varphi_1 \in \mathfrak{i}$. Therefore, we have:

Proposition 4.10. *Fixing a trivialisation*

$$t_1 : E|_{C_1} \to C_1 \times G,$$

with $t|_{c_1}(Q_1) = B \in (C_1 \times G)|_{c_1}/B = G/B$, as well as a trivialisation of L, identifies the space of Hecke transformations of (E, φ, Q) at $c_1 \in D$ up to equivalence with the affine Springer fibre

$$\mathrm{Fl}_G^{\varphi_1} := \{\sigma \in \mathrm{Fl}_G : \mathrm{Ad}_{\sigma^{-1}}(\varphi_1) \in \mathfrak{i}\} \subseteq \mathrm{Fl}_G$$

inside of the affine flag variety $\mathrm{Fl}_G = G((z))/\mathcal{I}$.

Choosing a different trivialisation $t'_1 = \eta \circ t_1$, where now $\eta \in \mathcal{I}$ (since t'_1 must also correspond Q_1 with the trivial parabolic structure B), results in a different space of Hecke transformations, since now $\varphi'_1 = \mathrm{Ad}_\eta(\varphi_1)$. As before, they are related by left multiplication by η.

$$\mathrm{Fl}_G^{\mathrm{Ad}_\eta(\varphi_1)} = \eta\, \mathrm{Fl}_G^{\varphi_1}.$$

Thus, without fixing any trivialisation we still have a well defined *type* of the Hecke transformation, which is an element in

$$\mathcal{I}\backslash G((z))/\mathcal{I} = \widetilde{W}_G.$$

Remark 4.11 *Suppose that the trivialisation t_1 that we fix does not send the parabolic structure to the trivial one, but rather $t|_{c_1}(Q_1) = gB \in (C_1 \times G)|_{c_1}/B = G/B$. Then, everything proceeds similarly, with the only difference that $\mathrm{Aut}(C_1 \times G, gB) = g\mathcal{I}g^{-1}$. Therefore the space of Hecke transformations is*

$$\mathrm{Fl}_{G,g}^{\varphi_1} = \{\sigma \in \mathrm{Fl}_{G,g} : \mathrm{Ad}_{\sigma^{-1}}(\varphi_1) \in \mathrm{Ad}_g(\mathfrak{i})\} \subseteq \mathrm{Fl}_{G,g} := G((z))/g\mathcal{I}g^{-1}.$$

If we change the previous trivialisation by $t_1' = \eta \circ t_1$, where $\eta \in G[[z]]$ (i.e. without forcing that the parabolic structure is mapped to the the same element), the relation between the spaces of Hecke transformations is as follows:

$$\mathrm{Fl}_{G,\eta(c_1)g}^{\mathrm{Ad}_\eta(\varphi_1)} = \eta\, \mathrm{Fl}_{G,g}^{\varphi_1}\, \eta(c_1)^{-1}.$$

Example 4.12 *Suppose $D = c$ for $c \in C$, structure group $G = \mathrm{PGL}_2(\mathbb{C})$, maximal torus T of diagonal matrices and Borel B of lower triangular matrices. Let $\beta = \beta_1$ denote the positive root. Take the following parabolic G-Higgs bundle, which is a fixed point of Borel type:*

$$E = \mathcal{O}_C \oplus (K_C(D))^{-1}(c)$$

$$\varphi = \begin{pmatrix} 0 & 0 \\ s_c & 0 \end{pmatrix}$$

$$Q = 0 \subsetneq (K_C(D))^{-1}|_c \subsetneq E|_c.$$

Here $s_c \in H^0(C, \mathcal{O}(c))$ denotes the defining section. Note that any choice of flag Q is compatible with φ since the residue vanishes at c. We pick as our trivialisation t_1 one that splits as a direct sum of a trivialisation of each line bundle summand. Then, with a Hecke transformation we may modify just the flag by using the (standard) Weyl group component of $\widetilde{W}$. For example, take

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This element is in the space of Hecke modifications, i.e. in the affine Springer fibre at $\varphi_1 = zX_\beta$. Indeed, $\mathrm{Ad}_{\sigma^{-1}}(zX_\beta) = zX_{-\beta} \in \mathfrak{i}$. It acts by permuting the transition functions of both line bundle summands, therefore the resulting bundle is

$$E' = (K_C(D))^{-1}(c) \oplus \mathcal{O}_C$$

$$\varphi' = \begin{pmatrix} 0 & s_c \\ 0 & 0 \end{pmatrix}$$

$$Q' = 0 \subsetneq (\mathcal{O}_C)|_c \subsetneq E'|_c.$$

Notice that only the parabolic structure Q' is changed, whereas E' and φ' are the same.

By using the cocharacter component of $\widetilde{W}$ we may change the vanishing multiplicities of our Higgs field. Indeed, take now the fundamental cocharacter

$$\sigma = z^{\omega^\vee} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

It is in the space of Hecke modifications, because $\mathrm{Ad}_{\sigma^{-1}}(zX_\beta) = z^{1+\beta(-\omega^\vee)}X_\beta = X_\beta \in \mathfrak{i}$. The resulting modification is

$$E' = \mathcal{O}_C \oplus (K_C(D))^{-1}$$
$$\varphi' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$Q' = Q.$$

Of course, we may perform modifications combining both operations if they are compatible. For example, if we take

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the product of the previous two transformations, it is not compatible since $\mathrm{Ad}_{\sigma^{-1}}(zX_\beta) = X_{-\beta} \notin \mathfrak{i}$. *Indeed, this is reflected by the fact that* $\mathcal{O}_C \oplus (K_C(D))^{-1}$ *with Higgs field* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *only admits one parabolic structure at* c *and it cannot be modified without breaking the compatibility with* φ. *Of course, we can also perform modifications by more general elements in* Fl_G *(not necessarily in* $\widetilde{W}$*) but these are not as easily expressible as the previous ones (they involve nontrivial extensions of the line bundles).*

Now we construct some explicit curves in Fl_G that showcase the closure relations in Proposition 4.5. As in the non-parabolic study, the main interest for this is that these particular curves are contained in certain affine Springer fibres, providing paths in the spaces of Hecke transformations explained above. Via the projection $\mathrm{Fl}_G \to \mathrm{Gr}_G$ we obtain similar curves that work outside of parabolic punctures.

We consider two main kinds of curves. We start by the simplest ones, which project to a point in $X_*(T) \subseteq \mathrm{Gr}_G$. These are of course well-known from the study of the classical Bruhat decomposition of G/B. Conceptually, they correspond to curves in the space of parabolic structures of a fixed meromorphic Higgs bundle.

Lemma 4.13. *Let* $w \in \widetilde{W}_G$ *and* $\beta \in \Delta^+$. *Let* $u_{-\beta} : \mathbb{G}_a \to G$ *be a root subgroup for* $-\beta$. *Then, the expression* $w \cdot u_{-\beta}(\lambda) \subseteq \mathrm{Fl}_G$, *for* $\lambda \in \mathbb{C}^\times$, *defines a curve whose closure is isomorphic to* $\mathbb{P}^1_{\mathbb{C}}$ *in* Fl_G, *with limits* w *as* $\lambda \to 0$ *and* ws_β *when* $\lambda \to \infty$.

Proof. It suffices to work in the subgroup of G corresponding to the $\mathfrak{sl}_2$-subalgebra for β. There,

$$u_{-\beta}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

It is clear that when $\lambda \to 0$ this limits to the identity. The other limit is computed by using

$$u_{-\beta}(\lambda) = \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The rightmost term lies in B so it vanishes in the quotient Fl_G, and the limit when $\lambda \to \infty$ is $s_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. □

Now we show that, whenever the endpoints lie in the affine Springer fibre Fl_G^e for the regular nilpotent element $e = \sum_{j=1}^r X_{\beta_j}$, the curve also does.

Lemma 4.14. *Let $e = \sum_{j=1}^r X_{\beta_j}$. Let $w = \mu \cdot u \in \widetilde{W}_G$ and $\beta \in \Delta^+$ be such that $l(us_\beta) = l(u) + 1$ and $u \leqslant us_\beta$ in the Bruhat order. Assume that $w, ws_\beta \in \mathrm{Fl}_G^e$. Then, the curve in Lemma 4.13 for w and β is also contained in Fl_G^e.*

Proof. Write $w = z^\mu \cdot u$ with $u \in W$. We know that μ is antidominant (recall Lemma 4.7) and then $w, ws_\beta \in \mathrm{Fl}_G^e$ is equivalent to $u^{-1}(\beta_p), s_\beta u^{-1}(\beta_p) \in \Delta^+$ for every $s_{\beta_p} \in J_\mu$. The statement of the Lemma follows by checking that

$$\mathrm{Ad}_{\exp(X_{-\beta})u^{-1}}(X_{\beta_p}) \in \mathfrak{b}$$

holds for every $s_{\beta_p} \in J_\mu$.

In order to show this, it suffices to prove that

$$u^{-1}(\beta_p) - k\beta$$

can never be a negative root when $k \geqslant 0$. Suppose otherwise, i.e. that $u^{-1}(\beta_p) - k\beta \in \Delta^-$ for some $k \geqslant 0$. Then, we must have $u^{-1}(\beta_p) + \beta \in \Delta^+$: if not, the root reflection s_β (which acts on the root string through $u^{-1}(\beta_p)$ with shift β by *flipping* it) would map $u^{-1}(\beta_p)$ to a negative root. Therefore, the root string through $u^{-1}(\beta_p)$ with shift β has length at least three. We start with the case where the length is exactly three. In other words, the root string consists of

$$-\gamma := u^{-1}(\beta_p) - \beta,$$

$$u^{-1}(\beta_p),$$

and

$$-s_\beta(\gamma) = u^{-1}(\beta_p) + \beta,$$

with $\gamma \in \Delta^+$.

Applying u, we have $-us_\beta(\gamma) = \beta_p + u(\beta)$. Note that $l(u) < l(us_\beta)$ implies $u(\beta) \in \Delta^+$, so we deduce $us_\beta(\gamma) \in \Delta^-$. This implies (e.g. [79, Section 5.6] applied to a reduced expression of us_β) that we can write

$$\gamma = s_{l_k} \dots s_{l_{j+1}}(\beta_{l_j})$$

for some $1 \leqslant j \leqslant k$, where

$$u = s_{l_1} \dots s_{l_{i-1}} s_{l_{i+1}} \dots s_{l_k}$$

is a reduced expression and

$$s_\beta = s_{l_k} \dots s_{l_{i+1}} s_{l_i} s_{l_{i+1}} \dots s_{l_k},$$

for simple root reflections $s_{l_i} = s_{\beta_{l_i}} \in S_W$. Note that here we use that the lengths of u and us_β differ by one.

On the other hand, applying u to the other endpoint of the root string gives $-u(\gamma) = \beta_p - u(\beta)$. Since β_p is simple (and therefore cannot be the sum of two positive roots) we deduce that $u(\gamma) \in \Delta^+$. From the reduced expression of u this implies that $j \leqslant i$. We claim that $i = j$, for if $j < i$ we can compute

$$s_\beta(\gamma) = (s_{l_k} \dots s_{l_{i+1}} s_{l_i} s_{l_{i+1}} \dots s_{l_k}) s_{l_k} \dots s_{l_{j+1}}(\beta_{l_j}) = s_{l_k} \dots \widehat{s_{l_i}} \dots s_{l_{j+1}}(\beta_{l_j})$$

and then, using the reduced expression of u and [79, Section 5.6] again, we deduce $s_\beta(\gamma) \in \Delta^+$, contradicting the fact that $s_\beta(\gamma) = -(u^{-1}(\beta_p) + \beta)$. However, this means that $\beta = \gamma$, so that $u^{-1}(\beta_p) = \beta - \gamma = 0$, a contradiction.

Having ruled out the case where the root string has length three, it suffices to address the case of length four. Root strings of this length only occur when one of the simple factors of $\mathfrak{g}$ is of type $\mathfrak{g}_2$, and there a manual check suffices: if β_1 denotes the short simple root and β_2 the long simple root, the Weyl group elements in the Springer fibre over X_{β_1} are

$$\{1, s_2, s_2s_1, s_2s_1s_2, s_2s_1s_2s_1, s_2s_1s_2s_1s_2\}$$

and for X_{β_2} they are

$$\{1, s_1, s_1s_2, s_1s_2s_1, s_1s_2s_1s_2, s_1s_2s_1s_2s_1\}.$$

In both cases, we see that the condition on the length (differing by one) forces β to be a simple root, but then $u^{-1}(\beta_p) - k\beta \in \Delta^-$ for some $k \geqslant 0$ forces the root string to be of the form $\{-\beta, 0, \beta\}$, which is of length three. □

Remark 4.15 *Lemma 4.14 implies that if $w = z^\mu \cdot u \in \mathrm{Fl}_G^e$ is such that u is not the longest element under the Bruhat order of ${}^{J_\mu}W$, then $\beta \in \Delta^+$ can be found such that the curve $wu_{-\beta}(\lambda)$ belongs to Fl_G^e and so does the limit $ws_\beta \geqslant w$. This is due to the* ***chain property*** *of ${}^{J_\mu}W$ [21, Theorem 2.5.5], one of which consequences is that any element different than the maximum has another element covering it in the Bruhat order, with length differing by one.*

The second kind of curve projects to a curve in Gr_G and involves shifting the cocharacter part by a coroot.

Proposition 4.16. *Let $z^\mu \cdot u \in \widetilde{W}_G$ and $\beta \in \Delta$. Let $u_\beta : \mathbb{G}_a \to G$ be a root subgroup for β. Let $m := \beta(\mu) + 1$. Then, the expression*

$$u_\beta(z^m\lambda^{-1})z^{\mu+\beta^\vee}s_\beta u \in G((z)),$$

for $\lambda \in \mathbb{C}^\times$, defines a curve in Fl_G with limit $z^\mu u$ as $\lambda \to 0$ and $z^{\mu+\beta^\vee}s_\beta u$ as $\lambda \to \infty$.

Proof. Since $u_\beta(0) = \mathrm{Id}$ it is clear that the limit when $\lambda \to \infty$ is the claimed one. Now we compute the other limit. Start by assuming $u = 1$. Note that the curve may be rewritten as

$$u_\beta(z^m\lambda^{-1})z^{\mu+\beta^\vee}s_\beta = z^\mu u_\beta(z^{m-\beta(\mu)}\lambda^{-1})z^{\beta^\vee}s_\beta = z^\mu u_\beta(z\lambda^{-1})z^{\beta^\vee}s_\beta.$$

This expression has the advantage that the factor multiplying z^μ on the right is fully contained in the subgroup of $G((z))$ corresponding to the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ associated to β. Thus, we may work in the copy of $\mathrm{SL}_2((z))$ mapping to said subgroup, and rewrite that term as

$$\begin{pmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & -z \\ z^{-1} & 0 \end{pmatrix}.$$

Now we can rewrite as follows:

$$\begin{pmatrix} \lambda^{-1} & -z \\ z^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda z^{-1} & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & -z \\ 0 & \lambda \end{pmatrix}.$$

The rightmost term can be disregarded in Fl_G since it is mapped inside of $\mathcal{I}$, therefore we see that the limit as $\lambda \to 0$ of the above expression is the identity, so the limit of the whole curve is z^μ.

If u is not trivial, we may observe the following:

$$u_\beta(z^m\lambda^{-1})z^{\mu+\beta^\vee}s_\beta u = uu_{u^{-1}(\beta)}(\pm z^m\lambda^{-1})z^{u^{-1}(\mu+\beta^\vee)}s_{u^{-1}(\beta)}$$

where the sign change in the root subgroup is not relevant for our purposes. Since now u is at the very left, it does not interfere with the right quotient by $\mathcal{I}$ and we may repeat the computations above (notice that $u^{-1}(\beta)(u^{-1}(\mu)) = \beta(\mu)$ so the multiplicity m does not change) to find that the limit at $\lambda \to 0$ is $uz^{u^{-1}(\mu)} = z^{\mu}u$. □

The remainder of this section is destined, as before, to investigating when these curves are contained in Fl_G^e.

Lemma 4.17. *Let $z^{\mu} \cdot u \in \widetilde{W}_G$ and $\beta \in \Delta$. Let $e = \sum_{i=1}^{r} X_{\beta_i} \in \mathfrak{g}$. Then, the curve in Proposition 4.16 is contained in Fl_G^e if and only if, for every simple root $\beta_i \in \Pi$, either*

$$-\mu(\beta_i) - \beta^{\vee}(\beta_i) - q_i > 0,$$

or

$$-\mu(\beta_i) - \beta^{\vee}(\beta_i) - q_i = 0 \ \text{ and } \ u^{-1}(s_{\beta}(\beta_i + q_i\beta)) \in \Delta^+,$$

where $q_i \geqslant 0$ is the greatest integer such that $\beta_i + q_i\beta \in \Delta$.

Proof. By definition of the affine Springer fibre, we just have to check when does

$$\mathrm{Ad}_{u^{-1}s_{\beta}z^{-\mu-\beta^{\vee}}u_{\beta}(-z^m\lambda^{-1})}(e)$$

belong to the standard Iwahori subalgebra $\mathfrak{i} = \mathrm{Lie}(\mathcal{I})$. For this, it suffices to check that every simple root vector X_{β_i} is mapped to $\mathfrak{i}$. Applying the first factor, we get

$$\mathrm{Ad}_{u_{\beta}(-z^m\lambda^{-1})}(X_{\beta_i}) = \sum_{j_i=0}^{q_i} c_{ij} z^{j_i m} X_{\beta_i + j_i\beta}$$

where the c_{ij} are nonzero complex numbers. Now we apply the second factor to each of those terms:

$$\mathrm{Ad}_{z^{-\mu-\beta^{\vee}}}(z^{j_i m} X_{\beta_i+j_i\beta}) = z^{j_i m - \mu(\beta_i+j_i\beta) - \beta^{\vee}(\beta_i+j_i\beta)} X_{\beta_i+j_i\beta}.$$

The valuation of the term above simplifies (using $\beta^{\vee}(\beta) = 2$ and $m = \beta(\mu) + 1$) to

$$j_i m - \mu(\beta_i) - j_i(m-1) - \beta^{\vee}(\beta_i) - 2j_i = -\mu(\beta_i) - \beta^{\vee}(\beta_i) - j_i.$$

Therefore, the minimum valuation that appears in the terms coming from X_{β_i} is $-\mu(\beta_i) - \beta^{\vee}(\beta_i) - q_i$. The two remaining operations (i.e. applying s_{β} and then w^{-1}) only permute the root vectors and do not change the valuations. The result follows by asking these to be positive or zero, noting that the latter requires the corresponding root vector to be in a positive root space. □

The conditions in the previous Lemma apply whenever μ is not minuscule and we have already arranged the part in ${}^{J_\mu}W$ to be *maximal* by following the curves of the first kind.

Lemma 4.18. *Let $z^{\mu} \cdot u \in \widetilde{W}_G \cap \mathrm{Fl}_G^e$ and suppose that w is the maximal element under the Bruhat order of ${}^{J_\mu}W$ and μ is not minuscule. Then, there exists $\beta \in \Delta^+$ such that $\mu + \beta^{\vee}$ is antidominant and the curve in Proposition 4.16 is contained in Fl_G^e.*

Proof. We construct β verifying the properties of Lemma 4.17. By Corollary 2.22 there exists $\beta^{\vee}$ such that for every $\beta_i \in \Delta^+$ we have

$$-\mu(\beta_i) - \beta^{\vee}(\beta_i) - q_i \geqslant 0.$$

Now we show that, for the chosen u, equality of the above expression automatically implies that $u^{-1}(s_\beta(\beta_i + q_i\beta)) \in \Delta^+$. Since $u \in {}^{J_\mu}W$ is the longest, we have:

$$u^{-1}(\beta_i) \in \Delta^+ \text{ for simple roots } \beta_i \in \Pi \text{ with } \mu(\beta_i) = 0$$

$$u^{-1}(\gamma) \in \Delta^- \text{ for positive roots } \gamma \in \Delta^+ \text{ with } \mu(\gamma) \neq 0.$$

Now observe that, when we apply s_β, we get

$$s_\beta(\beta_i + q_i\beta) = \beta_i - (q_i + \beta^\vee(\beta_i))\beta.$$

If equality holds in the first expression of the proof, that is, if $q_i + \beta^\vee(\beta) = -\mu(\beta_i)$, we have

$$s_\beta(\beta_i + q_i\beta) = \beta_i + \mu(\beta_i)\beta.$$

Finally, in order to determine whether $u^{-1}(\beta_i + \mu(\beta_i)\beta)$ is a positive root, we simply compute

$$\mu(\beta_i + \mu(\beta_i)\beta) = (1 + \mu(\beta))\mu(\beta_i).$$

Since $\mu + \beta^\vee$ is antidominant, we have $\mu(\beta) \leqslant -\beta^\vee(\beta) = -2$. As a consequence, the factor $1+\mu(\beta) \leqslant -1$ never vanishes. Therefore, it suffices to analyse these two cases:

- If $\mu(\beta_i) = 0$, then $u^{-1}(\beta_i + \mu(\beta_i)\beta) = u^{-1}(\beta_i) \in \Delta^+$.
- If $\mu(\beta_i) \neq 0$ then $\mu(\beta_i + \mu(\beta_i)\beta) \neq 0$ and $\beta_i + \mu(\beta_i)\beta$ is a negative root (β_i is simple, β positive and $\mu(\beta_i) \leqslant 0$) so $u^{-1}(\beta_i + \mu(\beta_i)\beta) \in \Delta^+$.

□

For later use we need the following shifted versions of the previous curves, which have the advantage of allowing for values of μ in the coweight lattice (even if they μ does not lift to T).

Lemma 4.19. *Let $w = \mu \cdot u \in \widetilde{W} = P_{\Delta^\vee} \rtimes W$ and $\beta \in \Delta$. Let $u_\beta : \mathbb{G}_a \to G$ be a root subgroup for β. Then, the expression $u_{-\beta}(\lambda)$ for $\lambda \in \mathbb{C}^\times$ defines a curve in Fl_G with limit 1 as $\lambda \to 0$ and s_β as $\lambda \to \infty$. Additionally, if u is not the maximal element under the Bruhat order of ${}^{J_\mu}W$, then there exists $\beta \in \Delta^+$ such that the previous curve is contained in $\mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)}$.*

Proof. This follows directly by taking the corresponding curve in Lemma 4.14, which can only be guaranteed to be defined when projected to the adjoint group $G_{\mathrm{ad}} = G/Z(G)$ (since $\widetilde{W} = \widetilde{W}_{G_{ad}}$), and then multiplying on the left by w^{-1}. The result belongs to $w^{-1}\,\mathrm{Fl}^e_{G_{\mathrm{ad}}} = \mathrm{Fl}^{\mathrm{Ad}_{w^{-1}}(e)}_{G_{\mathrm{ad}}}$ and, as can be seen from its expression, lifts to $\mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)} \subseteq \mathrm{Fl}_G$. □

Proposition 4.20. *Let $w = \mu \cdot u \in \widetilde{W} = P_{\Delta^\vee} \rtimes W$ and $\beta \in \Delta$. Let $u_\beta : \mathbb{G}_a \to G$ be a root subgroup for β. Let $m := \beta(\mu) + 1$. Then, the expression*

$$u_{u^{-1}(\beta)}(z^{m-\beta(\mu)}\lambda^{-1})z^{u^{-1}(\beta^\vee)}u^{-1}s_\beta u \in G((z)),$$

for $\lambda \in \mathbb{C}^\times$, defines a curve in Fl_G with limit 1 as $\lambda \to 0$ and $z^{u^{-1}(\beta^\vee)}u^{-1}s_\beta u$ as $\lambda \to \infty$. Additionally, if μ is antidominant, not minuscule and u is the maximal element under the Bruhat order of ${}^{J_\mu}W$, then there exists $\beta \in \Delta^+$ such that the previous curve is contained in $\mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)}$.

Proof. This follows directly by taking the corresponding curve in Lemmas 4.17 and 4.18 for the adjoint group $G_{\mathrm{ad}} = G/Z(G)$ and multiplying on the left by w^{-1}. The result belongs to $w^{-1}\,\mathrm{Fl}^e_{G_{\mathrm{ad}}} = \mathrm{Fl}^{\mathrm{Ad}_{w^{-1}}(e)}_{G_{\mathrm{ad}}}$ and, as can be seen from its expression, lifts to $\mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)} \subseteq \mathrm{Fl}_G$. □

4.2 The $\mathbb{C}^\times$-action on the space of parabolic Hecke transformations

Now we introduce the $\mathbb{C}^\times$-action on the space of Hecke transformations of $(E,\varphi,Q)\in\mathcal{M}_{sp}(G,\alpha)^{\mathbb{C}^\times}$ at a point $c\in C$ that corresponds with the $\mathbb{C}^\times$-action on $\mathcal{M}_{sp}(G,\alpha)$. We do so for all types of fixed point (E,φ,Q) (for any Vinberg $\mathbb{C}^\times$-pair $(G_0,\mathfrak{g}_1)$). Recall that $\zeta\in\mathfrak{g}_0$ denotes the grading element, which defines a cocharacter $\xi_\lambda\in G_{ad}$), and also recall that Hecke transformations occur at $c\in C$, C_1 is a formal disk around c and $C_0=C\smallsetminus\{c\}$.

There are two kinds of important local trivialisations at play: those aligned with the $(G_0,\mathfrak{g}_1)$-reduction and those aligned with the parabolic structure.

Definition 4.21 *Let $(E,\varphi,Q)\in\mathcal{M}_{sp}(G,\alpha)^{\mathbb{C}^\times}$ be a parabolic G-Higgs bundle fixed by the $\mathbb{C}^\times$-action reducing to a Vinberg pair $(G_0,\mathfrak{g}_1)$ and $c\in C$. A local trivialisation $t_1:E_1\to C_1\times G$ is called a G_0-**trivialisation** if $t_1(E_{G_0})=C_1\times G_0$. If, furthermore, we have $\varphi_1\in\mathfrak{g}_1[[z]]$, then it is called a $(G_0,\mathfrak{g}_1)$-**trivialisation**.*

*If $c\in D$, a local trivialisation $t_1:E_1\to C_1\times G$ is called **parabolic** if $t_1(Q_c)=B\in G/B$.*

Remark 4.22 *It is clear (since every principal bundle trivialises over a formal disk C_1) that the previous trivialisations always exist. However, in general it is not possible to have simultaneously a $(G_0,\mathfrak{g}_1)$-trivialisation which is parabolic at $c\in D$. Given a $(G_0,\mathfrak{g}_1)$-trivialisation, Theorem 1.92 shows that $t_1(Q_c)=g_0wB$ for some $g_0\in G_0$, $w\in W_{G_0}\backslash W$. In that case,*

$$t_1^p:=w^{-1}g_0^{-1}\cdot t_1:E_1\to C_1\times G$$

*defines a parabolic G_0-trivialisation. We say that it is **associated** to the $(G_0,\mathfrak{g}_1)$-trivialisation t_1.*

Recall that Proposition 4.10 identifies the space of Hecke transformations at $c\in D$ with $\mathrm{Fl}_G^{\varphi_1}$ using a parabolic trivialisation. Therefore, if we instead work with a $(G_0,\mathfrak{g}_1)$-trivialisation with $t_1(Q_c)=g_0wB$, the space becomes

$$\mathrm{Fl}^{\varphi_1}_{G,g_0w}=g_0w\left(\mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}g_0^{-1}}(\varphi_1)}\right)w^{-1}g_0^{-1}$$

according to Remarks 4.11 and 4.22.

Proposition 4.23. *Let $(E,\varphi,Q)\in\mathcal{M}_{sp}^{\mathbb{C}^\times}(G,\alpha)$ be a parabolic G-Higgs bundle fixed by the $\mathbb{C}^\times$-action and fix a $(G_0,\mathfrak{g}_1)$-trivialisation $t_1:E_1\to C_1\times G$ identifying the space of Hecke transformations with $\mathrm{Gr}_G^{\varphi_1}$ (if $c\notin D$, see Proposition 2.13) or with $\mathrm{Fl}^{\varphi_1}_{G,g_0w}$ (if $c\in D$, where $t_1(Q_c)=g_0wB\in G/B$, see Proposition 4.10 and the discussion above).*

There is a natural $\mathbb{C}^\times$-action in this space of Hecke transformations:

$$\sigma\mapsto\xi_\lambda^{-1}\sigma.$$

Denoting by (E',φ',Q') the Hecke transformation by σ and by $(E'_\lambda,\varphi'_\lambda,Q'_\lambda)$ the Hecke transformation by $\xi_\lambda^{-1}\sigma$, we have

$$(E',\lambda\varphi',Q')\simeq(E'_\lambda,\varphi'_\lambda,Q'_\lambda).$$

Proof. The proof follows verbatim that of Proposition 2.18, since the isomorphism $f_\lambda:(E,\varphi,Q)\simeq(E,\lambda\varphi,Q)$ does not change the local expression of the parabolic structures, and the fact that it preserves them implies that the map with local data $(f_\lambda^{-1},\mathrm{Id})$ constructed at the end of the proof also preserves the parabolic structures, which is the only extra check that was needed. □

Remark 4.24 *Note that if $c \in D$ and we take instead the associated parabolic G_0-trivialisation, the action on $\mathrm{Fl}_G^{\varphi_1}$ becomes $\sigma \mapsto w^{-1}g_0^{-1}\xi_\lambda^{-1}g_0 w\sigma = w^{-1}\xi_\lambda^{-1}w\sigma$.*

We can use Proposition 4.23 to show that the curves inside the Hecke transformation spaces studied in Lemma 4.13 and Proposition 4.16 are preserved by the $\mathbb{C}^\times$-action corresponding to $(G_0, \mathfrak{g}_1)$ given by the Borel grading.

Proposition 4.25. *Let $w = \mu \cdot u \in \widetilde{W}$ and $\beta \in \Delta^+$ such that $u \leqslant us_\beta$ in the Bruhat order of $\widetilde{W}$. There is an element $\sigma_{w,\beta} \in \mathrm{Fl}_G$ such that, for the $\mathbb{C}^\times$-action in Proposition 4.23 conjugated by u^{-1} (see Remark 4.24), we have*

$$\lim_{\lambda\to 0} \lambda \cdot \sigma_{w,\beta} = 1$$
$$\lim_{\lambda\to \infty} \lambda \cdot \sigma_{w,\beta} = s_\beta.$$

Proof. The element is that of Lemma 4.19, i.e. it is $u_{-\beta(\lambda)}(1)$ where $u_{-\beta} : \mathbb{G}_a \to G$ is the root subgroup for $-\beta$.

Indeed,

$$\lambda \cdot \sigma_{w,\beta} = u^{-1}\xi_\lambda^{-1}u \cdot u_{-\beta}(1) = (u^{-1}\xi)_\lambda^{-1}u_{-\beta}(1) = u_{-\beta}(\lambda^k)$$

where $k = \beta(u^{-1}(\zeta)) = u(\beta)(\zeta) > 0$, since $u(\beta) \in \Delta^+$ because $u \leqslant us_\beta$. □

Proposition 4.26. *Let $w = z^\mu \cdot u \in \widetilde{W}$ and $\beta \in \Delta^+$. There is an element $\sigma'_{w,\beta} \in \mathrm{Fl}_G$ such that, for the $\mathbb{C}^\times$-action in Proposition 4.23 conjugated by u^{-1} (see Remark 4.24), we have*

$$\lim_{\lambda\to 0} \lambda \cdot \sigma'_{w,\beta} = 1$$

$$\lim_{\lambda\to \infty} \lambda \cdot \sigma'_{w,\beta} = z^{u^{-1}(\beta^\vee)}u^{-1}s_\beta u.$$

Proof. Let $m := \beta(\mu) + 1$. This element is

$$\sigma'_{w,\beta} := u_{u^{-1}(\beta)}(z^{m-\beta(\mu)})z^{u^{-1}(\beta^\vee)}u^{-1}s_\beta u.$$

Indeed,

$$\begin{aligned}\lambda \cdot \sigma'_{w,\beta} &= \left(u^{-1}\xi_\lambda^{-1}u\right) u_{u^{-1}(\beta)}(z^{m-\beta(\mu)})z^{u^{-1}(\beta^\vee)}u^{-1}s_\beta u \\ &= u_{u^{-1}(\beta)}(\lambda^{-\mathrm{ht}(\beta)}z^{m-\beta(\mu)})z^{u^{-1}(\beta^\vee)}u^{-1}s_\beta u \cdot u^{-1}(s_\beta^{-1}(\xi_\lambda))\end{aligned}$$

The rightmost element belongs to $T \subseteq \mathcal{I}$. Therefore, we obtain (up to an orientation-preserving reparametrisation) the curve whose limits are described in Proposition 4.20. □

4.3 Classification of very stable parabolic Higgs bundles in terms of the twisted multiplicity divisor

For the remainder of the chapter we assume that fixed points are of Borel type. In this case, fixing a parabolic T-trivialisation and performing a Hecke transformation by an element in $\widetilde{W}_G \cap \mathrm{Fl}_G^{\varphi_1}$ results in a new fixed point (since $\widetilde{W}_G$ is fixed by the $\mathbb{C}^\times$-action in Proposition 4.23) of Borel type, and we can relate their multiplicity divisors.

Proposition 4.27. *Let $(E,\varphi,Q)\in\mathcal{M}_{sp}(G,\alpha)^{\mathbb{C}^\times}$ be a fixed point of Borel type and fix a parabolic T-trivialisation at $c\in D$ associated to some $(T,\mathfrak{g}_1)$-trivialisation. Let $w=z^\mu\cdot u\in \mathrm{Fl}_G^{\varphi_1}\cap\widetilde{W}_G$ and denote by (E',φ',Q') the corresponding Hecke transformation. It is again a fixed point of Borel type, with*

$$w_c(E',\varphi',Q')=w_c(E,\varphi,Q)\cdot w$$

and $w_p(E',\varphi',Q')=w_p(E,\varphi,Q)$ for $p\neq c$.

Proof. Denote $X:=\varphi_1\in\mathfrak{i}$ the local expression of the Higgs field in the fixed trivialisation. We abbreviate $\mu_c:=\mu_c(E,\varphi)$ and $u_c:=u_c(E,\varphi,Q)$. In the induced local expression for (E',φ',Q'), which is still parabolic, we have that the Higgs field is $\mathrm{Ad}_{u^{-1}z^{-\mu}}(X)\in\mathfrak{i}$. We also know that $\mathrm{Ad}_{u_c}(X)\in\mathfrak{g}_1[[z]]$ and $\mathrm{Ad}_{u'_cu^{-1}z^{-\mu}}(X)\in\mathfrak{g}_1[[z]]$. This means that the $u'_cu^{-1}z^{-\mu}u_c^{-1}$ must be a cocharacter (otherwise it would send $\mathrm{Ad}_{u_c}(X)$, which is of the form $\sum_{i=1}^r u_i(z)z^{k_i}X_{\beta_i}$ where $u_i(0)\neq 0$, outside of $\mathfrak{g}_1[[z]]$). We rewrite

$$u'_cu^{-1}z^{-\mu}u_c^{-1}=(z^{-u'_cu^{-1}(\mu)})u'_cu^{-1}u_c^{-1}.$$

Therefore, $u'_c=u_cu$ and the resulting cocharacter is $z^{-u_c(\mu)}$, so $\mu'_c=\mu_c-u_c(\mu)$. We can check in $\widetilde{W}$ that

$$w_c\cdot w=(\mu_c^{-1}u_c)(\mu\cdot u)=\mu_c^{-1}u_c(\mu)u_cu=(u_c(\mu)^{-1}\mu_c)^{-1}u_cu,$$

as desired. □

For completeness (although we will not need it) we note that the same result clearly holds for $c\notin D$ and $z^\mu\in\mathrm{Gr}_G^{\varphi_1}\cap X_*^-(T)$: it is Proposition 2.20.

Recall the notion of a reduced twisted multiplicity divisor from Definition 1.107. As in the non-parabolic setting, these reduced twisted multiplicity divisors will correspond to very stable upward flows. We already know that, in the simplest case where $w(E,\varphi,Q)=1$ from Example 1.114, the upward flow equals the Hitchin section and it is indeed very stable. Now we illustrate that if $w(E,\varphi,Q)$ is supported at D and reduced, we still get very stable sections of the Hitchin map as upward flows.

Example 4.28 *Assume that $G=G_{ad}$ is of adjoint type and suppose that (E,φ,Q) is a smooth fixed point of Borel type with twisted multiplicity divisor $w(E,\varphi,Q)$ supported at D (i.e. such that $w_c(E,\varphi,Q)=1$ whenever $c\notin D$). Assume that $l(w_c(E,\varphi,Q))=0$ for $c\in D$, see Proposition 1.7. We are going to prove that (E,φ,Q) is very stable by constructing its upward flow. We do so by performing a sequence of Hecke transformations at the parabolic punctures.*

We start at $c_1\in D$ with a local $(T,\mathfrak{g}_1)$-trivialisation $(C_{c_1}\simeq\operatorname{Spec}C[[z]],t_{c_1})$ around c_1 and perform the Hecke transformation by $\mu_{c_1}(E,\varphi)u_{c_1}(E,\varphi,Q)^{-1}\in\widetilde{W}=\widetilde{W}_G$. This is equivalent to performing the Hecke transformation by

$$u_{c_1}^{-1}(E,\varphi,Q)\left(\mu_{c_1}(E,\varphi)u_{c_1}(E,\varphi,Q)^{-1}\right)u_{c_1}(E,\varphi,Q)=w_{c_1}(E,\varphi,Q)^{-1}$$

in the associated parabolic T-trivialisation, so by Proposition 4.27 the resulting transformation (E^1,φ^1,Q^1) has

$$w_{c_1}(E^1,\varphi^1,Q^1)=w_{c_1}(E,\varphi,Q)w_{c_1}(E,\varphi,Q)^{-1}=1$$

and, since $1\in\mathrm{Fl}^e$, it is a well-defined parabolic G-Higgs bundle.

The transformation (E^1,φ^1,Q^1) comes with an induced local expression around c_1 which may no longer be $(T,\mathfrak{g}_1)$-trivialisation. However, we can change it by a left multiplication by $u_{c_1}(E,\varphi,Q)^{-1}$ in order to make it again a $(T,\mathfrak{g}_1)$-trivialisation at c_1, which is moreover parabolic at c_1. Of course, we can recover (E,φ,Q) by performing a Hecke transformation by $w_{c_1}(E,\varphi,Q)$ with this trivialisation, which reverses the whole process.

Now we iterate this at every puncture c_i, reaching an everywhere regular fixed point (E^s, φ^s, Q^s) with $w(E^s, \varphi^s, Q^s) = 1$. By Example 1.114, this fixed point is isomorphic to the canonical uniformising (E^0, φ^0, Q^0). Therefore, applying this isomorphism, we may assume that the local expression of the Higgs field in our local trivialisations around each c_i is constant, equal to $e = \sum_{j=1}^{r} X_{\beta_j}$.

An important remark is that, even after having applied this isomorphism, the Hecke transformations that revert back to (E, φ, Q) are still given by the elements $w_{c_i}(E, \varphi, Q)$ in the new local trivialisations. Indeed, a priori we can only guarantee that they have to be given by certain elements in the Schubert cell

$$\mathcal{I} \cdot w_{c_i}(E, \varphi, Q) \in \mathrm{Fl}_G .$$

However, by Proposition 4.5, since $l(w_{c_i}(E, \varphi, Q)) = 0$, this Schubert cell consists of a single point equal to $w_{c_i}(E, \varphi, Q)$.

To conclude, recall from the conclusion in Example 1.114 that the upward flow of (E^0, φ^0, Q^0) is given by the Hitchin section (E^0, φ^a, Q^0) for $a \in \mathcal{A}_G$. Since $\varphi^a(c_i) = \varphi^0(c_i)$ at every $c_i \in D$, locally we have

$$\varphi^a|_{C_{c_i}} = e + \sum_{j=1}^{r} a_j^i(z) \in \mathfrak{i},$$

where $a_j^i(z) \in z\langle f_j \rangle [[z]]$ is the local expression of the section $a_j f_j \in H^0(E^0(\mathfrak{g}) \otimes K_C(D))$ used to define φ^a. Thus, again by Proposition 4.5 and $l(w_{c_i}(E, \varphi, Q)) = 0$, which implies that $w_{c_i}(E, \varphi, Q)$ normalises $\mathcal{I}$, we may perform the Hecke modification by $w_{c_i}(E, \varphi, Q)$ at every c_i to (E^0, φ^a, Q^0) as well. Doing this in order (from c_s back to c_1) results in $(E, (\varphi^a)', Q)$, which define a section of the Hitchin map: locally around c_i (and after conjugating by $u_{c_i}(E, \varphi, Q)$ to make the resulting trivialisation a $(T, \mathfrak{g}_1)$-trivialisation) we have

$$(\varphi^a)'|_{C_{c_i}} = zX_{\beta_{j_0}} + \sum_{j \neq j_0} X_{\beta_j} + \sum_{j=1}^{r} (a')_j^i(z).$$

where j_0 is such that $\mu_{c_i}(E, \varphi) = \omega_{j_0}^\vee$, and $(a')_j^i(z) \in \mathfrak{g}[[z]]$ is given by $(a')_j^i(z) = \mathrm{Ad}_{z^{-\omega_{j_0}^\vee}} a_j^i(z)$. Note that $(a')_j^i(z)$ does not have poles as $\omega_{j_0}^\vee$ is minuscule, so its pairing with every root is at most one, while the $a_j^i(z)$ have at least valuation one. From the local expressions above it is clear that this section is in the upward flow of (E, φ, Q): applying the automorphism given by ξ_λ^{-1} we have

$$\mathrm{Ad}_{\xi_\lambda^{-1}}(\lambda \cdot (\varphi^a)'|_{C_{c_i}}) = zX_{\beta_{j_0}} + \sum_{j \neq j_0} X_{\beta_j} + \sum_{j=1}^{r} \lambda^{\deg(p_j)-1} (a')_j^i(z)$$

since every f_j has degree $1 - \deg(p_j)$ in the Borel grading, as explained in Example 1.114. Thus, the rightmost sum vanishes when $\lambda \to 0$.

Therefore, we get the subvariety

$$\mathcal{A}_{sp}(G) \simeq \{(E, (\varphi^a)', Q) : a \in \mathcal{A}_{sp}(G)\} \subseteq W^+_{(E,\varphi,Q)}.$$

Moreover, both sides are vector spaces of the same dimension (since (E, φ, Q) is smooth), hence equality holds. It then follows from this explicit description of the upward flow that (E, φ, Q) is very stable.

Example 4.29 *We now clarify the construction in the previous example in the simplest possible case. Let $G = \mathrm{PGL}_2(\mathbb{C})$, $D = c$ and $w_c(E, \varphi, Q) = (-\omega_1^\vee) \cdot s_{\beta_1}$. The corresponding fixed point, in terms of vector bundles, is isomorphic to*

$$E = \mathcal{O}_C \oplus K_C^{-1}$$

$$\varphi = \begin{pmatrix} 0 & 0 \\ s_c & 0 \end{pmatrix}$$
$$Q = \{E|_c \supseteq \mathcal{O}_C|_c \supseteq 0\},$$

where $s_c \in H^0(\mathcal{O}_C(c))$ is the canonical section. Given a point $a \in H^0(K_C^2(c))$ in the Hitchin base, the upward flow is given by keeping E and Q and using the Higgs fields

$$(\varphi^a)' = \begin{pmatrix} 0 & a \\ s_c & 0 \end{pmatrix}.$$

Now we verify that the curves in Propositions 4.25 and 4.26 can be used to obtain curves in $\mathcal{M}_{sp}(G,\alpha)^{\mathbb{C}^\times}$ by showing that, unless $w \in \widetilde{W} \cap \mathrm{Fl}^e_{G_{ad}}$ is minimal under the Bruhat order (i.e. unless it has length zero, see Proposition 1.7), we can always find curves of either of the two previous types, contained within $\mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)}$ and limiting at 1 when $\lambda \to 0$.

Proposition 4.30. *Let $w = \mu \cdot u \in \widetilde{W} \cap \mathrm{Fl}^e_{G_{ad}}$ and suppose that $l(w) \neq 0$. Then, there exists $\tilde{\sigma}_w \in \mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)}$ such that $\lambda \cdot \tilde{\sigma}_w \in \mathrm{Fl}_G^{\mathrm{Ad}_{w^{-1}}(e)}$ for all $\lambda \in \mathbb{C}^\times$ (for the $\mathbb{C}^\times$-action in Proposition 4.23 conjugated by u^{-1}, see Remark 4.24) and*

$$\lim_{\lambda \to 0} \lambda \cdot \tilde{\sigma}_w = 1.$$

Proof. Recall from Lemma 4.7 that μ is antidominant and $u \in {}^{J_\mu}W$ and from Proposition 1.7 that either μ is not minuscule or u is not the longest in ${}^{J_\mu}W$.

If u is maximal within ${}^{J_\mu}W$ but μ is not minuscule, by Proposition 4.20 we can find some positive root $\beta \in \Delta^+$ such that we can take $\tilde{\sigma}_w = \sigma'_{w,\beta}$ as in Proposition 4.26.

On the other hand, if u is not maximal within ${}^{J_\mu}W$, by Remark 4.15 there is another $u' = us_\beta \in {}^{J_\mu}W$ for some $\beta \in \Delta^+$ such that $u' \geqslant u$, and by Lemma 4.19 we can take $\tilde{\sigma}_w = \sigma_{w,\beta}$ as in Proposition 4.25. □

Now we consider the stability of the points in these curves, so that they are fully contained in the moduli space. If we start with a smooth (recall that for fixed points of Borel type this is equivalent to stable, by Proposition 1.105) fixed point with twisted multiplicity $w_c(E,\varphi,Q)$ and consider its Hecke transformations by the curve $\lambda \cdot \tilde{\sigma}_w$ having it as limit when $\lambda \to 0$, by openness and $\mathbb{C}^\times$-invariance of stability the result is stable for all $\lambda \in \mathbb{C}^\times$, and we just need to consider the other endpoint (when $\lambda \to \infty$). We do so for both kinds of curves.

Proposition 4.31. *Let $c \in D$ and let (E,φ,Q) be a fixed point of Borel type. Let $w_c = (E,\varphi,Q)$, $\mu_c = \mu_c(E,\varphi)$ and $u_c = u_c(E,\varphi,Q)$, so that $w_c = z^{-\mu_c} \cdot u_c \in \widetilde{W} \cap \mathrm{Fl}^e_{G_{\mathrm{ad}}}$. Fix a local parabolic T-trivialisation around $c \in D$ associated to a $(T,\mathfrak{g}_1)$-trivialisation, identifying the space of Hecke transformations with $\mathrm{Fl}_G^{\mathrm{Ad}_{w_c^{-1}}(e)}$ by Proposition 4.10. Let $\beta \in \Delta^+$ be such that $\sigma_{w_c,\beta} \in \mathrm{Fl}_G^{\mathrm{Ad}_{w_c^{-1}}(e)}$. Let (E',φ',Q') denote the Hecke transformation by*

$$\lim_{\lambda \to \infty} \lambda \cdot \sigma_{w_c,\beta} = s_\beta.$$

Then, if (E,φ,Q) is (semi)stable, so is (E',φ',Q').

Proof. We apply Proposition 1.106. Let $\chi \in X^*_+(T)$. Since $(E,\varphi) \simeq (E',\varphi')$ as meromorphic Higgs bundles, we have $d_E = d_{E'}$. Moreover, if $i \in \{1,\dots,s\}$ is the index such that $c = c_i$, we have by Proposition 4.27 that $\chi(u_{c_j}(E,\varphi,Q)(\alpha_j)) = \chi(u_{c_j}(E',\varphi',Q')(\alpha_j))$ for $j \neq i$, while

$$\begin{aligned}\chi(u_{c_i}(E',\varphi',Q')(\alpha_i)) &= \chi(u_c s_\beta(\alpha_i)) = \chi(u_c(\alpha_i)) - \beta(\alpha_i)\chi(u_c(\beta^\vee)) \\ &\leqslant \chi(u_c(\alpha_i)) = \chi(u_{c_i}(E,\varphi,Q)(\alpha_i))\end{aligned}$$

since $\beta(\alpha_i) \geqslant 0$, $u_c(\beta^\vee) \in \Delta_+^\vee$ (because $u_c s_\beta \geqslant u_c$) and χ is dominant. □

Proposition 4.32. *Let $c \in D$ and let (E,φ,Q) be a fixed point of Borel type. Let $w_c = (E,\varphi,Q)$, $\mu_c = \mu_c(E,\varphi)$ and $u_c = (E,\varphi,Q)$, so that $w_c = z^{-\mu_c}\cdot u_c \in \widetilde{W} \cap \mathrm{Fl}^e_{G_{\mathrm{ad}}}$. Fix a local parabolic T-trivialisation associated to a $(T,\mathfrak{g}_1)$-trivialisation around c, identifying the space of Hecke transformations with $\mathrm{Fl}_G^{\mathrm{Ad}_{w_c^{-1}}(e)}$. Let $\beta \in \Delta^+$ be such that $\sigma'_{w_c,\beta} \in \mathrm{Fl}_G^{\mathrm{Ad}_{w_c^{-1}}(e)}$. Let (E',φ',Q') denote the Hecke transformation by*

$$\lim_{\lambda\to\infty} \lambda\cdot\sigma'_{w_c,\beta} = z^{u_c^{-1}(\beta^\vee)} u_c^{-1} s_\beta u_c.$$

Then, if (E,φ,Q) is (semi)stable, so is (E',φ',Q').

Proof. We apply Proposition 1.106. Let $\chi \in X_+^*(T)$. By Proposition 4.27 we have $w_{c'}(E',\varphi',Q') = w_{c'}(E,\varphi,Q)$ for $c' \neq c$, while $w_c(E',\varphi',Q') = w_c(E,\varphi,Q)\cdot z^{u_c^{-1}(\beta^\vee)} u_c^{-1} s_\beta u_c$. Therefore,

$$\mu_c(E',\varphi') = \mu_c(E,\varphi) - \beta^\vee$$

$$u_c(E',\varphi',Q') = s_\beta u_c(E,\varphi,Q).$$

Thus,

$$\begin{aligned}\chi(d_{E'}) + \sum_{i=1}^{s}\chi(u_{c_i}(E',\varphi',Q')(\alpha_i)) &= \chi(d_E) - \chi(\beta^\vee) + \sum_{c'\in D\smallsetminus\{c\}}\chi(u_{c'}(E,\varphi,Q)(\alpha_{c'})) \\ &\quad + \chi(s_\beta u_c(\alpha_c)) \\ &= \chi(d_E) + \sum_{i=1}^{s}\chi(u_{c_i}(E,\varphi,Q)(\alpha_i)) \\ &\quad + \big(-\chi(\beta^\vee) + \chi(s_\beta u_c(\alpha_c) - u_c(\alpha_c))\big).\end{aligned}$$

Since (E,φ,Q) is stable, it suffices to prove that the last term $-\chi(\beta^\vee) + \chi(s_\beta u_c(\alpha_c) - u_c(\alpha_c))$ is not positive. Indeed:

$$-\chi(\beta^\vee) + \chi\big(s_\beta u_c(\alpha_c) - u_c(\alpha_c)\big) = -\chi(\beta^\vee) + \chi(-\beta(u_c(\alpha_c))\beta^\vee) = -(1+\beta(u_c(\alpha_c)))\chi(\beta^\vee).$$

Since χ is dominant and β positive, $\chi(\beta^\vee) \geqslant 0$. Moreover, since α_c is in the interior of the standard Weyl alcove, we have $\beta(u_c(\alpha_c)) \in (-1,1)$, so the term $1+\beta(u_c(\alpha_c))$ is also positive.

□

We are now in position to use all the analysis in the previous sections to establish the classification of very stable fixed points of Borel type. Recall the notion of a reduced twisted multiplicity divisor from Definition 1.107.

Theorem 4.33. *Let $(E,\varphi,Q) \in \mathcal{M}_{sp}(G,\alpha)$ be a smooth fixed point of Borel type. Then, (E,φ,Q) is very stable if and only if $w(E,\varphi,Q)$ is reduced.*

Proof. Suppose first that there is some $c \in C$ with $l(w_c(E,\varphi,Q)) \neq 0$. We may assume that $c \in D$, since the case with $c \notin D$ (which means that $\mu_c(E,\varphi)$ is not minuscule) can be found in Theorem

2.24, which essentially uses the projections of the curves $\sigma'_{w,\beta}$ to Gr_G, and works here identically due to the local nature of Hecke transformations.

Now, fix a local parabolic T-trivialisation for (E,φ,Q) around c associated to a $(T,\mathfrak{g}_1)$-trivialisation, identifying the space of Hecke transformations with $\mathrm{Fl}_G^{\mathrm{Ad}_{w_c(E,\varphi,Q)^{-1}}(e)}$ by Proposition 4.10. Let $(E_\lambda,\varphi_\lambda,Q_\lambda)$ for $\lambda\in\mathbb{C}^\times$ denote the Hecke transformations by $\lambda\cdot\tilde{\sigma}_{w_c(E,\varphi,Q)}$, where $\tilde{\sigma}_{w_c(E,\varphi,Q)}$ is that of Proposition 4.30. This is well defined since $\lambda\cdot\tilde{\sigma}_{w_c(E,\varphi,Q)}\in\mathrm{Fl}_G^{\mathrm{Ad}_{w_c(E,\varphi,Q)^{-1}}(e)}$. Moreover, since $\lim_{\lambda\to 0}\lambda\cdot\tilde{\sigma}_{w_c(E,\varphi,Q)}=1$, we have

$$\lim_{\lambda\to 0}(E_\lambda,\varphi_\lambda,Q_\lambda)\simeq(E,\varphi,Q).$$

By openness and $\mathbb{C}^\times$-invariance of stability, every $(E_\lambda,\varphi_\lambda,Q_\lambda)$ is stable. Moreover, the bundle

$$(E_\infty,\varphi_\infty,Q_\infty):=\lim_{\lambda\to\infty}(E_\lambda,\varphi_\lambda,Q_\lambda)$$

is also stable by Propositions 4.31 and 4.32. The whole curve and its limits must lie in $(h_G^{sp})^{-1}(0)$ (since Hecke transformations preserve the Hitchin fibre) so they are nilpotent and, in particular, strongly parabolic (see Remark 1.93). This means that (E,φ,Q) is wobbly, as the $(E_\lambda,\varphi_\lambda,Q_\lambda)$ for $\lambda\in\mathbb{C}^\times$ are nilpotent in its upward flow.

For the converse, start by assuming that $G=G_{ad}$ is of adjoint type. Assume that all the $w_c(E,\varphi,Q)$ have zero length. The proof is by induction on the size of the support of $w(E,\varphi,Q)$, and it is very similar to that of Theorem 2.24 but using the techniques developed in this chapter. The base case corresponds to empty support, in which case the construction in Example 1.114 shows that it is very stable and its upward flow is a section of h_G^{sp}. Otherwise, take $c\in C$ with $w_c(E,\varphi,Q)\neq 1$. Choose a $(T,\mathfrak{g}_1)$-trivialisation for this $c\in C$. Assume that it is not very stable. Then, there is $(E',\varphi',Q')\in(h_G^{sp})^{-1}(0)\cap W^+_{(E,\varphi,Q)}$. By Proposition 1.113 the Higgs field can be locally expressed as

$$\varphi'_1=\varphi_1+\sum_{\beta\in\Delta^-}f_\beta(z)X_\beta+\sum_{j=1}^{r}f_j(z)\beta_j^\vee,$$

where the coefficients f_β, f_j are in $\mathbb{C}[[z]]$. Therefore, if $c\notin D$, the proof of Theorem 2.24 continues identically to conclude that (E,φ,Q) is very stable, by performing a Hecke transformation by the minuscule coweight $\omega_j^\vee$ such that $z^{\omega_j^\vee}\in\mathrm{Gr}_G^{\varphi_1}$, which therefore also satisfies $z^{\omega_j^\vee}\in\mathrm{Gr}_G^{\varphi'_1}$, to reduce the size of the support of the twisted multiplicity divisor and apply the induction hypothesis. We may then assume that $w(E,\varphi,Q)$ is supported at D. But then the explicit construction of the upward flow in Example 4.28 shows that (E,φ,Q) is very stable.

Finally, if G is not of adjoint type, the converse follows by using the map

$$\mathcal{M}(G,\alpha)\to\mathcal{M}(G_{\mathrm{ad}},\alpha)$$

given by $(E,\varphi,Q)\mapsto(E/Z(G),\varphi,Q)$. This preserves stability and the twisted multiplicity divisor, as well as the $\mathbb{C}^\times$-action and the image via the Hitchin map. Therefore, if $w(E,\varphi,Q)$ were reduced but

$$W^+_{(E,\varphi,Q)}\cap(h_G^{sp})^{-1}(0)$$

had nonzero dimension, applying this map would yield that

$$W^+_{(E/Z(G),\varphi,Q)}\cap h^{-1}_{G^{sp}_{\mathrm{ad}}}(0)$$

would also have nonzero dimension with the same twisted multiplicity divisor, a contradiction. □

4.4 Mirror line bundle for parabolic very stable upward flows

Let (E,φ,Q) be a smooth fixed point of Borel type with twisted multiplicity divisor $w := w(E,\varphi,Q)$ supported at D. In this case, we know (cf. Example 4.28) that the corresponding upward flow $W^+_{(E,\varphi,Q)}$ is a section of the Hitchin map h^{sp}_G. Therefore, the general philosophy of mirror symmetry being realised (generically over the Hitchin base) by a fibrewise Fourier–Mukai transform explained in Section 1.3 dictates that we should be able to produce a *mirror* line bundle over $\mathcal{M}_{sp}(G^\vee,\alpha)$ recording how $W^+_{(E,\varphi,Q)}$ shifts from a given Hitchin section $W^+_{(E^0,\varphi^0,Q^0)}$.

We propose a natural candidate for such a mirror line bundle. Suppose there exists a universal parabolic bundle

$$\mathbb{E} \to \mathcal{M}_{sp}(G^\vee,\alpha)\times C,$$

that is, a $G^\vee$-bundle endowed with universal parabolic structures

$$\mathbb{Q}_i \in H^0(\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c_i\}, \mathbb{E}|_{\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c_i\}}/B^\vee)$$

such that $(\mathbb{E},\mathbb{Q})|_{\{(E,\varphi,Q)\}\times C} \simeq (E,Q)$. Then, we have a reduction $(E_{\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c_i\}})_{B^\vee}$ to the Borel subgroup $B^\vee \subseteq G^\vee$ via $\mathbb{Q}_i$ and, given a cocharacter $\mu \in X_*(T) = X^*(T^\vee) = X^*(B^\vee)$, we can construct a line bundle $\mu((\mathbb{E}_{\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c_i\}})_{B^\vee})$. Recall that our chosen twisted multiplicity divisor w is supported over D, therefore at every c_i we have a naturally associated antidominant minuscule cocharacter $\mu_i^{-1} := \mu_{c_i}(E,\varphi)^{-1}$.

Conjecture 4.34. The line bundle on $\mathcal{M}_{sp}(G^\vee,\alpha)$ mirror to the structure sheaf $\mathcal{O}_{W^+_{(E,\varphi,Q)}}$ on $\mathcal{M}_{sp}(G,\alpha)$ is given by

$$\mathcal{L}_w := \bigotimes_{c_i\in D} \mu_i^{-1}((\mathbb{E}_{\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c_i\}})_{B^\vee})$$

up to tensoring by a fixed line bundle (independent of w) given by the chosen normalisation for $\mathbb{E}$.

There are a few remarks to be taken into consideration. Firstly, we are implicitly assuming that every $\mu_i \in P_{\Delta^\vee}$ lifts to $X^*(T^\vee)$. This is automatic if the group $G = G_{\mathrm{ad}}$ is of adjoint type, but for other groups it imposes an additional restriction.

Secondly, this is dependent on the existence of $\mathbb{E}$ as a bundle over $\mathcal{M}_{sp}(G^\vee,\alpha)\times C$. Of course, the proposed bundles always exist at the level of the moduli stack (see [92]), but on the moduli space they may only descend with a twist by a $Z(G)$-gerbe corresponding to a class in $H^2(\mathcal{M}_{sp}(G^\vee,\alpha), Z(G))$ (see e.g. [70, Section 6.2.1]). Even if $\mathbb{E}$ does not exist globally over the moduli space, the specific combination of μ_i may cause the induced twisting for $\mathcal{L}_w$ to become trivial, but this is not automatic.

Lastly, in order to check this conjecture using the duality via Fourier–Mukai transforms along the fibres of the Hitchin map, it is convenient to have a *good* description of said fibres (ideally, as honest abelian varieties or their torsors) generically over the Hitchin base. In other words, it is convenient to have smoothness of the corresponding spectral or cameral curve. However, even if we have chosen our parabolic weights α in the interior of the standard Weyl alcove, for a generic point over the Hitchin base the strongly parabolic condition makes $\varphi|_{c_i}$ into a regular *nilpotent* element of $\mathfrak{g}$. Therefore, the cameral curve is generally not smooth, complicating the approach for general G.

Therefore, in the remainder of this section we verify Conjecture 4.34 in the most favourable setting of $G = \mathrm{PGL}_n(\mathbb{C})$ and $G^\vee = \mathrm{SL}_n(\mathbb{C})$ with weights α (always in the interior of the standard alcove) chosen generically so that $\mathcal{M}_{sp}(\mathrm{SL}_n(\mathbb{C}),\alpha)$ is smooth. In this way, all μ_i^{-1} make sense as characters of $T^\vee$, the universal bundle $\mathbb{E}$ exists globally, uniquely up to normalisation (see [17, Theorem 1.1] for

the vanishing of the twisting class, or the explanation in [18, Section 4]. Note, however, that in this case the universal bundle might not be an $\mathrm{SL}_n(\mathbb{C})$-bundle in the sense that it might not be possible to trivialise its determinant over the whole of $\mathcal{M}_{sp}(\mathrm{SL}_n(\mathbb{C}),\alpha)\times C$, since the relevant gerbe class that is shown to vanish is considered in $H^2(\mathcal{M}_{sp}(\mathrm{SL}_n(\mathbb{C}),\alpha),\mathbb{C}^\times)$ and not in the corresponding group with coefficients in $Z(\mathrm{SL}_n(\mathbb{C}))=\mathbb{Z}/n\mathbb{Z}$) and, as we review in the next subsection, the generic fibres of the Hitchin map are abelian varieties via the Beauville–Narasimhan–Ramanan (BNR) correspondence, as the generic spectral curves are smooth.

Having restricted to the case $G=\mathrm{PGL}_n(\mathbb{C})$, in the sequel we work in the usual setting of vector bundles with full flags at the parabolic points, as described in Example 1.95 (for $\mathrm{PGL}_n(\mathbb{C})$, the only change needed for $\mathrm{SL}_n(\mathbb{C})$ is that E becomes an actual vector bundle, no longer up to line bundle tensoring, and is subject to the constraint $\det(E)\simeq\mathcal{O}_C$).

Let us recall the aspects of the BNR correspondence that will be needed in order to check the duality. In our setting with full flags it turns out to be largely similar to the classical BNR correspondence in [8]. We follow [57, Section 3], see also [121] for a more general description involving any flag types.

Let $\mathcal{A}:=\mathcal{A}_{sp}(\mathrm{SL}_n(\mathbb{C}))\simeq\mathcal{A}_{sp}(\mathrm{PGL}_n(\mathbb{C}))=\bigoplus_{j=2}^n H^0(C,K_C^j(j-1)D)$ denote the Hitchin base and choose a point $a=(a_2,\dots,a_n)\in\mathcal{A}$. Let $S:=\mathrm{Tot}(K_C(D))=\underline{\mathrm{Spec}}\,\mathrm{Sym}^*(K_C^{-1}(-D))$ denote the total space of $K_C(D)$, equipped with the map $\pi_S:S\to C$. There is a tautological section $x\in H^0(S,\pi_S^*(K_C(D)))$, as well as pull-backs $a_j':=\pi_S^*(a_j)\in H^0(S,\pi_S^*(K_C^j((j-1)D)))$. We map these naturally into $H^0(S,\pi_S^*(K_C^j(jD)))$, resulting in a well defined section

$$s_a:=x^n+a_2'x^{n-2}+\cdots+a_n'$$

in $H^0(S,\pi_S^*(K_C^n(nD)))$.

Definition 4.35 *The **spectral curve** associated to a is the closed subscheme of S defined by $s_a=0$, and it is denoted by C_a. In other words, it is the cover:*

$$\pi_a:C_a:=\underline{\mathrm{Spec}}(\mathrm{Sym}^*(K_C^{-1}(-D))/\mathcal{I}_a)\to C$$

where $\mathcal{I}_a$ is the ideal sheaf in $\mathrm{Sym}^*(K_C^{-1}(-D))$ generated by the elements of the form $\nu+\nu a_2+\cdots+\nu a_n$ with ν a local section of $K_C^{-n}(-nD)$.

An important aspect of C_a in our strongly parabolic setting is that it is totally ramified over each $c_i\in C$, as every a_j vanishes at the parabolic punctures. We denote by $c_i^a\in C_a$ the only closed point with $\pi_a(c_i^a)=c_i$. Even then, there is an open dense subset $\mathcal{A}^s\subseteq\mathcal{A}$ such that $a\in\mathcal{A}^s$ implies that C_a is smooth, see [57, Lemma 3.1].

Proposition 4.36 (BNR correspondence, [57, Lemma 3.2]). *Given a point $a\in\mathcal{A}^s$, its fibre $(h^{sp}_{\mathrm{SL}_n(\mathbb{C})})^{-1}(a)$ is isomorphic to*

$$P_a^\vee:=\{L\in\mathrm{Pic}(C_a):\det((\pi_a)_*L)\simeq\mathcal{O}_C\},$$

and the fibre $(h^{sp}_{\mathrm{PGL}_n(\mathbb{C})})^{-1}(a)$ is isomorphic to

$$P_a:=\mathrm{Pic}(C_a)/\pi_a^*\,\mathrm{Pic}(C).$$

Remark 4.37 *The notation P_a and $P_a^\vee$ only seeks to convey that the former corresponds to G and the latter to $G^\vee$. Strictly speaking, they are not dual abelian varieties: $P_a^\vee$ is a torsor for the Prym variety $\mathrm{Prym}(C_a/C)$, while P_a decomposes in n components which are torsors for the identity component, the latter being an honest abelian variety dual of $\mathrm{Prym}(C_a/C)$.*

The corresponding point (E_L, φ_L, Q_L) in the fibre is recovered from the line bundle $L \to C_a$ by taking $E_L := (\pi_a)_* L$ as the vector bundle and $\varphi_L := (\pi_a)_* x$ as the Higgs field. Since this Higgs field is regular nilpotent at each $c_i \in D$, by Lemma 1.1 there is already a unique parabolic structure $Q_{L,i}$ at each c_i which is compatible with φ_L, so it is automatically determined without any extra data. We can describe it explicitly as

$$Q^j_{L,i} := (\pi_a)_* \left(\frac{L(-(n-j)c_i^a)}{L(-nc_i^a)} \right).$$

In particular, note for later use that

$$E_L/Q^j_{L,i} = (\pi_a)_* \left(\frac{L}{L(-(n-j)c_i^a)} \right) = (\pi_a)_* L|_{(n-j)c_i^a}.$$

Now we set up the appropriate universal bundles required to test the duality. Let

$$\pi_{\mathscr{C}} : \mathscr{C} \to \mathcal{A}^s \times C$$

denote the family of spectral curves, so that $\mathscr{C}|_{\{a\}\times C} \simeq C_a$. In this way, we can describe $(h^{sp}_{\mathrm{SL}_n(\mathbb{C})})^{-1}(\mathcal{A}^s) \subseteq \mathcal{M}_{sp}(\mathrm{SL}_n(\mathbb{C}), \alpha)$ as

$$\mathscr{P}^\vee := \{\mathscr{L} \in \mathrm{Pic}(\mathscr{C}/\mathcal{A}^s) : [\det((\pi_{\mathscr{C}})_* \mathscr{L})] = [\mathcal{O}_{\mathcal{A}^s \times C}] \text{ in } \mathrm{Pic}(\mathcal{A}^s \times C/\mathcal{A}^s)\},$$

which is a torsor for the relative Prym given by $\mathrm{Prym}(\mathscr{C}/\mathcal{A}^s \times C)$, and $(h^{sp}_{\mathrm{PGL}_n(\mathbb{C})})^{-1}(\mathcal{A}^s)$ as the quotient of relative Picard group schemes $\mathscr{P} := \mathrm{Pic}(\mathscr{C}/\mathcal{A}^s)/\pi^*_{\mathscr{C}}(\mathcal{A}^s \times \mathrm{Pic}(C))$.

As indicated in Remark 4.37, we expect a duality of torsors over dual abelian varieties, therefore according to the general SYZ philosophy, certain gerbes (sometimes referred to as *B-fields* following the physical terminology behind mirror symmetry) should enter the picture (see [73, Section 1]), accounting for the glueing of the chosen local trivialisations of the torsors. However, in our current parabolic setting we can avoid this entirely, precisely because the existence of the very stable upward flows that we are currently considering: these are sections of the Hitchin map which provide a well-defined choice of origin, identifying P_a and each component of $P_a^\vee$ with the abelian varieties themselves in a coherent way across the whole of $\mathcal{A}^s$. This absence of the necessity for B-fields was already observed in [63, Theorem 3.5]. We thus let

$$\mathscr{O} : \mathcal{A}^s \to \mathscr{P}$$

be the section given by $\mathscr{O}(a) := \mathcal{O}_{C_a}(c_1^a)$ (through the BNR correspondence, this gives precisely the very stable upward flow determined by $\mu(E, \varphi) = \omega_1^\vee \cdot c_1$). By taking tensor powers, this choice determines an origin for every component of the fibres of $h^{sp}_{\mathrm{PGL}_n(\mathbb{C})}$ across $\mathcal{A}^s$.

Similarly, we need to choose such an origin

$$\mathscr{O}^\vee : \mathcal{A}^s \to \mathscr{P}^\vee$$

for $h^{sp}_{\mathrm{SL}_n(\mathbb{C})}$. If $\deg(D)$ is even, this can be achieved straightforwardly by taking M to be a square root of $K_C(D)$ and setting $\mathscr{O}^\vee(a) := \pi_a^*(M^{n-1})$: this merely corresponds to one of the upward flows with trivial multiplicity divisor $\mu(E, \varphi) = 1$. Otherwise, no such upward flows exist, so one needs to select a different very stable upward flow, or any section of $h^{sp}_{\mathrm{SL}_n(\mathbb{C})}$. This can always be done, as explained in the proof of [63, Theorem 3.5].

We now fix a universal line bundle

$$\mathbb{L} \to \mathscr{P}^\vee \times_{\mathcal{A}^s} \mathscr{C}$$

rigidified with the following normalisations:

$$\mathbb{L}|_{\mathscr{C}^\vee(\mathcal{A}^s)\times_{\mathcal{A}^s}\mathscr{C}} = \mathcal{O}_{\mathscr{C}}, \qquad \mathbb{L}|_{\mathscr{P}^\vee\times_{\mathcal{A}^s}\{c_1^a\}_{a\in\mathcal{A}_s}} = \mathcal{O}_{\mathscr{P}^\vee}.$$

Being universal, it satisfies that for $M \in \mathrm{Prym}(C_a/C)$, we have $\mathbb{L}|_{\{M\}\times C_a} \simeq M$. Moreover, over $\mathcal{A}^s$ we can recover (the normalisation that we will consider of) the universal parabolic bundle $\mathbb{E}$ via the BNR correspondence, that is

$$\mathbb{E}|_{(h^{sp}_{SL_n(\mathbb{C})})^{-1}(\mathcal{A}^s)\times C} \simeq (\mathrm{Id}\times\pi_{\mathscr{C}})_*\mathbb{L}.$$

Similarly, the filtration $\mathbb{Q}_i$ of $\mathbb{E}|_{(h^{sp}_{SL_n(\mathbb{C})})^{-1}(\mathcal{A}^s)\times\{c_i\}}$ can be recovered via the BNR correspondence as explained before.

The duality for the Hitchin systems over $\mathcal{A}^s$ is given by the Fourier–Mukai transform, which requires introducing a Poincaré line bundle on the product of our two abelian varieties. However, since we only care about the dual line bundles of sections

$$\mathscr{S} : \mathcal{A}^s \to \mathscr{P},$$

we may bypass this and write it down merely in terms of $\mathbb{L}$.

Remark 4.38 *The dual of the structure sheaf of the image of a section $\mathscr{S} : \mathcal{A}^s \to \mathscr{P}$ which is induced by a line bundle $\mathscr{L} \to \mathscr{C}$ can be computed as follows. If $\mathbb{P} \to \mathscr{P}\times_{\mathcal{A}^s}\mathscr{P}^\vee$ denotes a relative Poincaré line bundle, the desired dual is the line bundle over $\mathscr{P}^\vee$ given by the slice $\mathbb{P}|_{\mathscr{S}(A^s)\times_{A_s}\mathscr{P}^\vee}$. Said restriction can be related to $\mathbb{L}$ [98, Equation 5-18] as follows:*

$$\mathbb{P}|_{\mathscr{S}(A^s)\times_{A_s}\mathscr{P}^\vee} = \det\left(R(p_{\mathscr{P}^\vee})_*\left(\mathbb{L}\otimes p_{\mathscr{C}}^*(\mathscr{L})\right)\otimes\det(R(p_{\mathscr{P}^\vee})_*\mathbb{L}\right)^{-1}.$$

The rightmost factor accounts for the correct normalisation in a way that computing the mirror of the structure sheaf of our basepoint section $\mathscr{O}$, which corresponds to the line bundle $\mathscr{L}$ with divisor given by the preimage of $\mathcal{A}^s\times\{c_1\}$ in $\mathscr{C}$, gives $\mathcal{O}_{\mathscr{P}^\vee}$.

We now verify Conjecture 4.34 for the structure sheaf of $W_w^+ := W^+_{(E,\varphi,Q)}$. The first step is to identify the line bundles corresponding to W_w^+ via the BNR correspondence. In order to do this, it is useful to have an explicit description of W_w^+ in terms of vector bundles, and we can do so by recalling Example 4.28. Let j_i be the index such that $\mu_i = \omega_{j_i}^\vee$, denote by

$$D_j := \sum_{c_i\in D : j_i = j} c_i$$

and, for notational consistency, set $D_n := D - D_1 - \cdots - D_{n-1}$ (similarly, set $j_i = n$ for those $c_i \in D_n$).

For $a \in A$, we denote by (E,φ_a,Q) the point in $W_w^+ \cap (h^{sp}_{\mathrm{PGL}_n(\mathbb{C})})^{-1}(a)$. Then, we have

$$E := \bigoplus_{j=0}^{n-1} K_C^{-j}(-jD + \sum_{k=1}^{j} D_k),$$

$$\begin{aligned}\varphi_a|_{K_C^{-j}(-jD+\sum_{k=1}^j D_k)} &:= s_{D_{j+1}} \in H^0(\mathcal{O}_C(D_{j+1})) \\ &= H^0\Bigg(\mathrm{Hom}\left(K_C^{-j}\left(-jD+\sum_{k=1}^{j} D_k\right),\right. \\ &\qquad \left. K_C^{-(j+1)}\left(-(j+1)D+\sum_{k=1}^{j+1} D_k\right)\otimes K_C(D)\right)\Bigg),\end{aligned}$$

for $j < n-1$, where s_{D_j} denotes the canonical section, and

$$\varphi_a|_{K_C^{-(n-1)}(-(n-1)D+\sum_{k=1}^{n-1} D_k)} := \sum_{j=2}^{n} a_j,$$

where each summand is viewed as

$$\begin{aligned} a_j \in H^0\Bigg(\mathrm{Hom}\Bigg(& K_C^{-(n-1)}\left(-(n-1)D + \sum_{k=1}^{n-1} D_k\right), \\ & K_C^{-(n-j)}\left(-(n-j)D + \sum_{k=1}^{n-j} D_k\right) \otimes K_C(D)\Bigg)\Bigg) \\ = H^0\left(K_C^j\left(jD - \sum_{k=n-j+1}^{n-1} D_k\right)\right) \end{aligned}$$

under the natural map of $H^0(K_C^j(j-1)D)$ into the right hand side above (since the sum of all D_j is an effective divisor lower or equal than D).

The parabolic structure is given explicitly by the permutation indicated by $u(E,Q)$. Set $V_j := K_C^{-j}(-jD + \sum_{k=1}^{j} D_k)$. Then, for $c_i \in D_j$, the corresponding structure is

$$\begin{aligned} Q_{c_i} := 0 \subsetneq V_{j-1}|_{c_i} \subsetneq V_{j-2}|_{c_i} \oplus V_{j-1}|_{c_i} \subsetneq \cdots \subsetneq & \bigoplus_{l=0}^{j-1} V_l|_{c_i} \\ \subsetneq & \bigoplus_{l=0}^{j-1} V_l|_{c_i} \oplus V_{n-1}|_{c_i} \\ \subsetneq & \bigoplus_{l=0}^{j-1} V_l|_{c_i} \oplus V_{n-2}|_{c_i} \oplus V_{n-1}|_{c_i} \\ \subsetneq & \cdots \subsetneq E|_{c_i}. \end{aligned}$$

Using this explicit description, we compute the corresponding spectral line bundle.

Proposition 4.39. *Let $a \in \mathcal{A}^s$ and let $(E,\varphi_a,Q) \in \mathcal{M}_{sp}(\mathrm{PGL}_n(\mathbb{C}),\alpha)$ denote the point $W_w^+ \cap (h^{sp}_{\mathrm{PGL}_n(\mathbb{C})})^{-1}(a)$. Then, under the identification of Proposition 4.36, (E,φ_a,Q) corresponds to the line bundle*

$$L^w := \bigotimes_{c_i \in D} \mathcal{O}_{C_a}((n-j_i)\cdot c_i^a),$$

where $j_i \in \{1,\dots,n-1\}$ is such that $\mu_i = \omega_{j_i}^\vee$.

Proof. We compute $(\pi_a)_*(L^w)$ locally. Trivialising $K_C(D)$ around c_i (thus $\pi_S^* K_C(D)$ around c_i^a), we find that the tautological section x defines a uniformising parameter around c_i^a, since the smoothness of C_a implies that $\mathrm{ord}_{c_i^a}(a_n') = n \cdot \mathrm{ord}_{c_i}(a_n) = n$.

We may choose as well a uniformiser z at c_i, such that we get identifications of the completed local rings $R = \widehat{\mathcal{O}_{C,c_i}} \simeq \mathbb{C}[[z]]$, $R' := \widehat{\mathcal{O}_{C_a,c_i^a}} \simeq \mathbb{C}[[x]] = \mathbb{C}[[z]][x]/(x^n + a_2(z)x^{n-2} + \cdots + a_n(z))$ where z equals x^n up to multiplication by an invertible element. We choose $\{1,x,\dots,x^{n-1}\}$ as a basis for $\widehat{\mathcal{O}_{C_a,c_i^a}}$ over $\mathbb{C}[[z]]$. Around c_i, local sections of $\pi_* L^w$ correspond to meromorphic functions

$$f = \sum_{j=0}^{n-1} f_j(z)x^j \in \widehat{\mathcal{O}_{C_a,c_i^a}}$$

with $\operatorname{ord}_{c_i^a}(f) \geqslant j_i - n$. Since $\operatorname{ord}_{c_i^a}(f_j(z)x^j) = j + n\operatorname{ord}_{c_i}(f_j)$, which are all distinct for $0 \leqslant j \leqslant n-1$, the condition is equivalent to

$$j + n\operatorname{ord}_{c_i}(f_j) \geqslant j_i - n,$$

that is,

$$\operatorname{ord}_{c_i}(f_j) \geqslant -\left(1 + \frac{j - j_i}{n}\right).$$

Therefore, the f_j are allowed a simple pole at c_i whenever $j \geqslant j_i$, so that globally

$$\pi_* L^w = \bigoplus_{j=0}^{n-1} K_C^{-j}(-jD + \sum_{k=1}^{j} D_k) = E.$$

It is clear from the above analysis that $(\pi_a)_*(x) = \varphi_a$, and there is a unique parabolic structure compatible with φ_a since smoothness of C_a implies that it is regular nilpotent at the parabolic punctures. □

We are now in position of computing the mirror line bundle.

Proposition 4.40. *For $G = \mathrm{PGL}_n(\mathbb{C})$ and generic weights α in the interior of the standard alcove, the dual of the structure sheaf $\mathcal{O}_{W_w^+}$ where w is a twisted multiplicity divisor supported at D equals the line bundle $\mathcal{L}_w$ proposed in Conjecture 4.34 over $\mathcal{A}^s$.*

Proof. We choose the universal bundle $\mathbb{E}$ normalised as before. Let $\mathbb{E}_i := \mathbb{E}|_{h^{-1}_{\mathrm{SL}_n(\mathbb{C})}(\mathcal{A}^s)\times\{c_i\}}$, equipped with its filtration

$$0 \subsetneq \mathbb{Q}_i^1 \subsetneq \cdots \subsetneq \mathbb{Q}_i^n = \mathbb{E}_i.$$

According to Proposition 4.39, it suffices to verify that the dual of $\mathscr{L} := \mathcal{O}_{\mathscr{C}}((n-j_i)\cdot c_i^\bullet)$, where $c_i^\bullet = \pi_{\mathscr{C}}^{-1}(\mathcal{A}^s \times \{c_i\})$, equals (up to fixed normalisation)

$$\omega_{j_i}^{-1}(\mathbb{E}_i) = \det(\mathbb{E}_i/\mathbb{Q}_i^{j_i}).$$

Thus, by Remark 4.38 it suffices to show

$$R(p_{\mathscr{P}^\vee})_*\mathbb{L}|_{\mathscr{P}^\vee \times_{\mathcal{A}^s}\{(n-j_i)\cdot c_i^\bullet\}} \simeq \mathbb{E}_i/\mathbb{Q}_i^{j_i},$$

which is a direct consequence of the description of the filtration as a pushforward via the BNR correspondence, as explained after Proposition 4.36. □

Remark 4.41 *Mirror symmetry predicts that, since W_w^+ is a holomorphic Lagrangian subvariety of $\mathcal{M}_{sp}(\mathrm{PGL}_n(\mathbb{C}), \alpha)$ (a BAA-brane), the dual line bundle should carry a hyperholomorphic connection (a BBB-brane). From the algebraic construction it is only apparent that the one proposed in Conjecture 4.34 is holomorphic in one of the structures, so an interesting question is to check their hyperholomorphicity. This could be achieved, as in [70, Remark 7.1], by considering the moduli space as a hyperkähler reduction and checking that the proposed line bundles can be obtained from unitary representations of the corresponding gauge group.*

Remark 4.42 *Due to the local behaviour of Hecke transformations, if the divisor w is not supported at D, the dual can be obtained as in [70, Section 6]. In other words, for the very stable upward flow with twisted multiplicity coweight w, letting $\mu_c := \mu_c(E, \varphi)$ and considering the representations $\rho^{\mu_c} : G^\vee \to \mathrm{GL}(V^{\mu_c})$ with highest weight μ_c, the mirror bundle for $\mathcal{O}_{W_w^+}$ should be (up to normalisation) given by*

$$\bigotimes_{c\notin D} \rho^{\mu_c}(\mathbb{E}|_{\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c\}}) \otimes \bigotimes_{c_i\in D} \mu_i^{-1}((\mathbb{E}|_{\mathcal{M}_{sp}(G^\vee,\alpha)\times\{c_i\}})_{B^\vee}).$$

Future work

Having classified very stable Higgs bundles with generically regular Higgs field both in $\mathcal{M}(G)$ and $\mathcal{M}_{sp}(G,\alpha)$, as well as started the program of the classification with non-regular Higgs field in $\mathcal{M}(G)$, we now outline future directions of work concerning this circle of ideas.

Completing the classification for non-regular fixed points

There are several considerations regarding how to extend the classification to other non-regular fixed points of the $\mathbb{C}^\times$-action. The first one is to execute the same approach of Chapter 3 to arbitrary gradings

$$\mathfrak{g} = \mathfrak{g}_{1-m} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{m-1}$$

in order to obtain more wobbly components. There are two main considerations that are needed when steering away from Hermitian gradings:

1. The centre of G_0 could (and, for most gradings, will) have a dimension greater than one. This implies that generally there are more characters $\chi : \mathfrak{g}_0 \to \mathbb{C}$ from which other topological invariants independent of τ can be constructed.
2. General gradings do not have an associated real form $G_{\mathbb{R}} \subseteq G$ of Hermitian type so that the Cayley correspondence for $\mathcal{M}(G_{\mathbb{R}})$ can be exploited. A priori only the Cayley correspondence for $\mathcal{M}(G_0, \mathfrak{g}_1)$ from Theorem 1.63 is available.

The reason why the first point is relevant is that, while Lemma 3.6 provides a technique for constructing nilpotent Higgs fields in the upward flow of a given fixed point (E,φ) of any type, this technique requires a refined control of certain invariants. For example, the same strategy of Theorem 3.8 could be executed by controlling $\deg E(\mathfrak{g}'_{1-m})$, where $\mathfrak{g}'_{1-m} \subseteq \mathfrak{g}_{1-m}$ is the subspace defined by taking the $\mathrm{ad}(h)$-eigenspaces with eigenvalue different than $2-2m$ (recall that (e,h,f) is a $\mathfrak{sl}_2(\mathbb{C})$-triple where $e \in \Omega_\varphi \subseteq \mathfrak{g}_1$ is any element in the G_0-orbit taken generically by φ, $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$). More generally, subspaces $\mathfrak{g}_j^k$ (where k is the $\mathrm{ad}(h)$-eigenvalue) such that $k > 2j$ and $j < 0$ can be used.

The previous approach allows to write a more general (i.e. for any type) version of Theorem 3.8 provided that one can bound $\deg E(\mathfrak{g}_{1-m})$ for a JM-regular grading (this bound would then be applied to the quantity $\deg(E_\sigma(\widehat{G}_0)(\mathfrak{g}_{1-m}^{2-2m}))$ that appears in the proof). This is no longer a story about the Toledo character, since the Toledo invariant is (up to a constant) given by $\sum_{j>0} j \deg(E(\mathfrak{g}_j))$ which in general does not determine $\deg E(\mathfrak{g}_{1-m})$. However, one approach we propose is leveraging the **trace characters** $\chi_j : \mathfrak{g}_0 \to \mathbb{C}$ defined by

$$\chi_j(X) := \mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{g}_j}).$$

For JM-regular gradings these characters are endowed, like the Toledo invariant, with a non-trivial **relative invariant** $F_j : \mathfrak{g}_1 \to \mathbb{C}$ defined by

$$F_j(X) := \det((\mathrm{ad}(X)|_{\mathfrak{g}_{-j}})^{2j}).$$

It has the key property that $F_j(gX) = \tilde{\chi}_j(g)F_j(X)$ for $g \in G_0$ and $X \in \mathfrak{g}_1$, where $\tilde{\chi}_j$ lifts χ_j (up to a constant) to a character of G_0. Therefore, considering the evaluation $F_j(\varphi)$ results in a bound for $\deg E(\mathfrak{g}_{1-m})$ in the JM-regular setting, as in the original proof of the Arakelov–Milnor inequality [14, Theorem 5.3].

However, using this bound to prove the more general wobbliness theorem for non-JM-regular gradings would result in a condition on the invariant $\deg E(\mathfrak{g}_{1-m})$. The missing ingredient is then to study this invariant, in particular whether the stability of (E, φ) allows to produce an Arakelov–Milnor-type bound for it in order to determine whether the condition is reasonable or otherwise empty. More generally this leads to the study of the space of topological invariants for the G_0-bundle in the presence of the Higgs field φ making the pair stable, which still is ongoing work in progress with García-Prada. For $G = \mathrm{GL}_n(\mathbb{C})$, this study has been fully carried out in work of García-Prada, Heinloth and Schmitt [53, 54].

The second point would be required to generalise the wobbliness result for JM-regular gradings of Theorem 3.5, and it has several exciting approaches. On one hand, while there is not an associated Hermitian form $G_{\mathbb{R}} \subseteq G$, we can associate a **Vinberg $\mathbb{Z}/m\mathbb{Z}$-pair** (see [49]) $(G_0, \mathfrak{g}_1 \oplus \mathfrak{g}_{1-m})$ which has an associated moduli space $\mathcal{M}(G_0, \mathfrak{g}_1 \oplus \mathfrak{g}_{1-m})$ parametrising **cyclic Higgs bundles**. This moduli space also has an Arakelov–Milnor inequality for the Toledo invariant defined similarly on the G_0-bundle, as well as a Cayley correspondence. This was developed in recent work of García-Prada and the author [50] and could be exploited to identify more wobbly components, although it comes with some difficulties such as the *Cayley partner* pair not having a clear structure (it may not be a Vinberg pair in general, which complicates the geometry of the resulting partner moduli space).

Another approach is to make use of the Cayley map for magical $\mathfrak{sl}_2(\mathbb{C})$-triples developed in [25]. This map can be constructed for JM-regular gradings such that the associated $\mathfrak{sl}_2(\mathbb{C})$-triple is *magical* (this is a technical condition, such pairs were classified in loc. cit.). This kind of gradings include the Hermitian ones. Since the source of the Cayley map also includes a moduli space twisted by a line bundle of high degree (some higher power of K_C), it can be used to claim wobbliness of the fixed points on its image in a similar way as Theorem 3.5.

Besides these generalisations of Chapter 3, the following step is to develop new techniques allowing to further extend the classification, already starting with the Hermitian case. In this regard, proving that a fixed point component does not contain very stable fixed points can be done by assuming any property that holds generically in the component (since being very stable is an open condition, by the upper semicontinuity of the fibre dimension of the limit map $h_G^{-1}(0) \to \mathcal{M}(G)^{\mathbb{C}^\times}$ between proper varieties, whose fibres are precisely the intersections $h_G^{-1}(0) \cap W^+_{(E,\varphi)}$ whose zero-dimensionality is equivalent to being very stable) and exploiting this property to construct a nilpotent Higgs bundle in the upward flow.

Understanding generic properties on a given fixed point component can be accomplished by deforming the stability condition on $(G_0, \mathfrak{g}_1)$-Higgs pairs. As explained in [14, Definition 4.11], for $(G_0, \mathfrak{g}_1)$-Higgs pairs one can factor in a stability parameter with values in the centre of the maximal compact subalgebra of $\mathfrak{g}_0$ and study how the moduli spaces change upon modifying this parameter. The advantage of this is that the moduli space typically stays generically the same, and for certain values of the parameter the corresponding stable Higgs pairs are much more rigid. This *wall crossing* technique is very common when dealing with objects of this kind.

For example, for a non-JM-regular grading one can modify the stability parameter so that the Higgs field is fully contained in the open orbit $\Omega \subseteq \mathfrak{g}_1$, which allows to construct nilpotent Higgs field using similar arguments as those for the Cayley correspondence. Unfortunately, the condition

for wobbliness that stems from the naive application of this seems to be more restrictive than the one of Theorem 3.8.

On the other hand, for a JM-regular grading, the stability parameter can be modified so that the Higgs field is generically in the open orbit, but not everywhere. This seems to reflect that, as in the Borel type case, there is also a local component to this problem that needs to be considered. One possible program, motivated by noticing that sometimes Hecke transformations can be performed in order to bring the Higgs field into the open orbit at the remaining finitely many points, is to attempt the construction of nilpotent elements in the upward flow via Hecke transformations as in the case of Borel type. This naturally leads to the study of non-principal $\mathbb{C}^\times$-actions on Gr_G (as described in Proposition 2.18) and on affine Springer fibres for non-regular nilpotent orbits, but exactly how has still to be determined. These actions behave differently as for example the fixed point locus is no longer discrete, but it is given as the affine Grassmannian of G_0 itself. A related approach consists of starting with well-understood fixed points, similar to the uniformising Higgs bundles (for example, those modelled on more general Slodowy slices other than the Kostant section) and try to perform Hecke transformations using the previously mentioned $\mathbb{C}^\times$-action in order to obtain more wobbly Higgs bundles. We intend to pursue this in the near future.

All of these discussions concern proving wobbliness, but certainly at present one of the most interesting questions is to find more examples of very stable Higgs bundles, which is typically a much harder question, as developing techniques to prove that a fixed point is very stable is still an ongoing issue. Even if one is able to show that the corresponding upward flows in the given affine Springer fibre in Gr_G for the $\mathbb{C}^\times$-action above are zero-dimensional (which was essentially the strategy in the Borel case), one should not expect this to imply that the fixed point is very stable. As explained above, for non-Borel gradings the problem has certain global component which cannot be explained by local modifications. Indeed, even within our results for Hermitian gradings one can check that usually the limiting point in the closure of the upward flow has a different fixed point type, which would not be the case with our notion of Hecke transformations (which preserve the fixed point type). This was not a problem in the regular setting only because all the relevant dynamics were confined in components of Borel type.

Finally, another related further direction is to study how to extend the classification in the non-regular setting for $\mathcal{M}_{sp}(G,\alpha)$. For parabolic Higgs bundles one can also consider the definition of the Toledo invariant, as in [55, Section 4], [90] or [15, Theorem 8.2], and exploit it to find wobbly Higgs bundles using a similar approach.

Mirror symmetry aspects

Another direction to be pursued after the present work is the clarification of some of the mirror symmetry aspects involving very stable Higgs bundles, for example the identification of the mirror BBB-branes. As explained in Remark 4.42, the dual brane of a very stable upward flow for the Higgs bundle (E,φ) of Borel type with multiplicity divisor $\mu(E,\varphi)$ should be given by the tensor product

$$\bigotimes_{c\in C} \rho^{\mu_c(E,\varphi)}(\mathbb{E}|_{\mathcal{M}(G^\vee)\times\{c\}})$$

for a universal bundle $\mathbb{E}\to\mathcal{M}(G^\vee)\times C$ and the representation $\rho^{\mu_c(E,\varphi)}$ of $G^\vee$ with highest weight $\mu_c(E,\varphi)$. Viewing very stable upward flows as Hecke transformations from a uniformising Higgs bundle allows to use the intertwining [39, Theorem 4.13] of Hecke operators and tensorisation operators (given by the previous universal bundles) via the Fourier–Mukai transform duality to establish that the above is the dual brane at least at the level of stacks.

However, it would be desirable to more concrete evidences of this duality at the level of moduli spaces, similar to those presented in [70, Section 6] or Section 4.4. One of the difficulties (which was not present in the parabolic setting, at least for groups of type A) is the existence of the universal

$G^\vee$-bundle itself, which is only guaranteed when G is simply connected. In general, the bundle can only be interpreted as an object twisted by a $Z(G)$-gerbe, but still the vector bundle proposed above may lift if enough, suitable factors are tensored with each other. In those cases it would be interesting to test the duality via fibrewise Fourier–Mukai transform.

Another issue which is present when G is not simply connected is that $\mu_c(E,\varphi)$ belongs in general to the coweight lattice $P_{\Delta^\vee}$ but it may not land in $X_*(T)$, so the representation $\rho^{\mu_c(E,\varphi)}$ may not be well defined. In these cases it would be interesting to understand how to construct the dual brane. The simplest example of this would be to compute the dual brane of the upward flow of the $\mathrm{SL}_2(\mathbb{C})$-Higgs bundle given by

$$E = L^{-1} \oplus L,$$
$$\varphi = \begin{pmatrix} 0 & 0 \\ s_c s_{c'} & 0 \end{pmatrix}$$

where $c \neq c'$ are two distinct points in C and L is a square root of $K_C^{-1}(c+c')$. This fixed point cannot be obtained as a sequence of Hecke transformations from a uniformising section of the Hitchin map (without exiting $\mathcal{M}(\mathrm{SL}_2(\mathbb{C}))$), but rather as a *simultaneous* Hecke transformation at both $c, c' \in C$.

Another pending issue is that mentioned at Remark 4.41 about rigorously showing that the mirror line bundle computed in Section 4.4 has a hyperholomorphic structure and a hyperholomorphic connection. This would involve techniques from differential geometry, most likely studying the construction of this bundle via the hyperkähler reduction procedure and relating it to representations of the gauge group compatible with all Kähler structures.

Lastly, regarding parabolic moduli spaces $\mathcal{M}(G,\alpha)$ it would be interesting to consider other symplectic leaves where $\mathbb{C}^\times$ acts by scaling. This would require considering more general parabolic weights as such symplectic leaves are, roughly speaking, parametrised by nilpotent orbits on a Levi subgroup of the parabolic corresponding to α, so that whenever this parabolic equals the Borel $B \subseteq G$ the only possibility is the zero orbit corresponding to the strongly parabolic leaf.

At any rate, the above question is interesting in order to understand what the mirror partner for a given symplectic leaf of $\mathcal{M}(G,\alpha)$ should be. It was somehow clear that the strongly parabolic moduli spaces for G and $G^\vee$ should be mirror to each other, but for more general leaves this requires understanding how to pair the nilpotent orbits obtained from G with those of $G^\vee$, which is not obvious. In general, an important pending direction is to establish the duality of parabolic Hitchin systems in a similar generality as [39].

Other considerations

Finally, we comment on two additional problems related to the techniques and results of this thesis. The first one is to consider finite subgroups $\mathbb{Z}/m\mathbb{Z} \subseteq \mathbb{C}^\times$, generated by a primitive m-th root of unity $\zeta_m \in \mathbb{C}^\times$, and restrict the problem to the fixed point locus $\mathcal{M}(G)^{\zeta_m}$. Naturally, the $\mathbb{C}^\times$-action restricts to this locus and we have $\mathcal{M}(G)^{\mathbb{C}^\times} \subseteq \mathcal{M}(G)^{\zeta_m}$, but $h_G^{-1}(0)^{\zeta_m}$ becomes smaller than $h_G^{-1}(0)$. Therefore, there could be wobbly fixed points (E,φ) without nilpotent, ζ_m-fixed points in their upward flows, leading to a notion of **m-very stable** fixed points.

One motivation for considering such notion is that, whenever $m = 2$, the resulting fixed point locus $\mathcal{M}(G)^{-1}$ is a Lagrangian subvariety and defines the support of a BAA-brane, corresponding to the locus of $G_{\mathbb{R}}$-Higgs bundles for certain real forms $G_{\mathbb{R}} \subseteq G$ (called of *Hodge type*), as explained in [56] and Section 1.3. Therefore, the -1-fixed locus of the upward flow of a fixed point represents the intersection of two Lagrangian subvarieties, which is a relevant consideration the study of the Fukaya category of $\mathcal{M}(G)$ and mirror symmetry.

In [61], Hausel and the author classified 2-very stable Higgs bundles in the generically regular Higgs field case for $G = \mathrm{GL}_n(\mathbb{C})$ (analogously for $\mathrm{PGL}_n(\mathbb{C})$), yielding a similar result to Theorem 2.24 but in terms of the multiplicity divisor being 2-*reduced*, meaning that all of its coefficients are 2-minuscule, that is, none of them can be decomposed as the sum of a dominant coweight and any positive number of positive coroots of even height. Unlike minuscule coweights, there are typically infinitely many 2-minuscule ones. For example, for $G = \mathrm{GL}_2(\mathbb{C})$ every fixed point of Borel type is 2-minuscule.

Interestingly, in loc. cit. it was shown that the finite cover $h_G|_{W_{\mathcal{E}}^{+,-1}}$ for some choices of 2-very stable $\mathcal{E}$ can be described in terms of the equivariant cohomology rings of a certain *dual* symmetric space of the Nadler dual group [100] for the quasi-split real form $G_{\mathbb{R}} \subseteq G$ of Hodge type. This phenomenon was further explored in [43].

The generalisation to arbitrary groups and arbitrary values of m can be approached with the techniques of Chapter 2. First, note that the curves in Proposition 2.19 are fixed by ζ_m if and only if m divides the height of β. Therefore, if the multiplicity coweight at $c \in C$ can be decomposed as the sum of a dominant coweight and a positive coroot of height divisible by m, and if the curve corresponding to said coroot satisfies Lemma 2.21, the m-wobbliness follows. There are two issues:

- As seen in Corollary 2.22 and its proof, not every positive coroot satisfies the conditions of Lemma 2.21. We circumvented this by showing that those coroots realising covering relations in the Bruhat order could always be used. For simply laced groups any coroot can in fact be selected, but otherwise some more careful analysis is needed, e.g. working only with covering relations in the corresponding Bruhat order generated by the lattice of coroots of height divisible by m.
- While the usual Bruhat order is generated by $\mu \leqslant \lambda$ if and only if $\lambda - \mu$ is a positive coroot, this is not the case for this generalised order in terms of m. In other words, a coweight μ could be m-minuscule yet all the differences $\mu - \beta$ for β a positive coroot of height divisible by m could be non-dominant: not all the covering relations can be realised by coroots themselves. Again, the best hope is to look at the covering relations of the induced Bruhat order and work with those elements of the root lattice as the *shifts* realising m-wobbliness.

For the converse, proving that if all multiplicity coweights are m-minuscule the corresponding fixed point is m-very stable requires studying the m-fixed points of the space of Hecke transformations, which can be rephrased in terms of root heights and the previous m-minuscule condition by studying the weight decomposition of the tangent space to the Schubert varieties, described for example in [134, Proposition 2.1.5]. In short, there is enough evidence that a generalisation of the work in [61] can be accomplished, and we intend to pursue this in the future.

Besides the case of fixed points of Borel type, it is worth noting that the wobbliness results of Chapter 3, namely Theorems 3.5 and 3.8, actually imply 2-wobbliness which is stronger than wobbliness. This is because the nilpotent Higgs bundle in the upward flow constructed in both proofs is fixed by the action of -1.

Finally, the second additional question is more related to the geometry of the affine Grassmannian. In short, the $\mathbb{C}^\times$-action on Gr_G which is natural for the present work (in Proposition 2.18) is missing a very common ingredient: the loop rotation given by scaling the parameter z. Combining the scaling by the principal cocharacter with this loop rotation results in a better behaved action (for example, its upward flows are finite-dimensional, while this was not the case for our action, where we would only obtain finite-dimensional upward flows after restricting to a Schubert variety). This is interesting because, for example, the resulting equivariant cohomology for affine Schubert varieties under this action can be seen as some sort of *quantisation* of the one for our action, see e.g. [69, Theorem 3.1].

Therefore, a natural question is whether a Higgs bundle moduli space context where this action naturally emerges can be naturally constructed. Recalling how the $\mathbb{C}^\times$-action in Proposition 2.18 was obtained, such a context should involve a $\mathbb{C}^\times$-action on the base curve C, which can only happen if $C = \mathbb{P}^1$ and therefore requires the presence of punctures. These punctures would have to be contained in two points c_0, c_∞ for them to be preserved under a $\mathbb{C}^\times$-action. This implies, in order for $K_C(D)$ to have positive degree, that at least one of the punctures should impose higher order vanishing on the Higgs field, leading to the setting of moduli spaces for parabolic or meromorphic Higgs bundles with irregular singularities (see e.g. [13, 45, 46]). We intend to apply our techniques to the study of such a $\mathbb{C}^\times$-action in this setting.

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