Classification of very stable Higgs bundles

Miguel González

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Very stable Higgs bundles, equivariant multiplicity and mirror symmetry

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Abstract We define and study the existence of very stable Higgs bundles on Riemann surfaces, how it implies a precise formula for the multiplicity of the very stable components of the global nilpotent cone and its relationship to mirror symmetry. The main ingredients are the Bialynicki-Birula theory of $\mathbb{C}^*$-actions on semiprojective varieties, $\mathbb{C}^*$ characters of indices of $\mathbb{C}^*$-equivariant coherent sheaves, Hecke transformation for Higgs bundles, relative Fourier–Mukai transform along the Hitchin fibration, hyperholomorphic structures on universal bundles and cominuscule Higgs bundles.

Contents

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Definition (Hausel–Hitchin, 2022)

A smooth fixed point $\mathcal{E}$ is **very stable** if and only if $W_{\mathcal{E}}^{+}$ is **closed**. Otherwise, **wobbly**.

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- **Very stable** Higgs bundles $\leftrightarrow$ **finite cover** (finite, flat, surjective, generically unramified).

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- **Very stable** Higgs bundles $\leftrightarrow$ $\mathbb{C}^\times$-invariant, **closed, affine** irreducible complex Lagrangian subvarieties.

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- G connected semisimple algebraic group over $\mathbb{C}$, Lie algebra $\mathfrak{g}$.
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- $\mathcal{M}(G)$ parametrises isomorphism classes of **polystable** G-Higgs bundles.
- **Example.** $G = \mathrm{PGL}_n(\mathbb{C}) \to$ pair of **rank** n **vector bundle** E (up to line bundle tensoring) and $\varphi : E \to E \otimes K_C$ traceless.

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- $\lambda \in \mathbb{C}^\times$ acts: $(E, \varphi) \mapsto (E, \lambda\varphi)$.

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- In general, theory of **Vinberg $\mathbb{C}^\times$-pairs**, representations of Levi subgroups $G_0 \subseteq G$.

Fixed points and gradings

$\mathbb{Z}$-gradings:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

with $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$.
$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0 \rightsquigarrow G_0 \subseteq G$ connected subgroup.
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Prop. (Simpson 1988, Biquard–Collier–García-Prada–Toledo 2023)

Fixed points $(E, \varphi) \in \mathcal{M}(G)^{\mathbb{C}^\times}$ characterised by:

- E reduces to E_{G_0} a G_0-bundle.
- $\varphi \in H^0(E_{G_0}(\mathfrak{g}_j) \otimes K_C)$ for $j \neq 0$

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Theorem (Hausel–Hitchin 2022)

(E, φ) very stable if and only if $\varphi_{n-1} \circ \cdots \circ \varphi_1 : L_0 \to L_{n-1} \otimes K_C^{n-1}$ is a reduced divisor (no multiple zeroes).

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- $G = \mathrm{GL}_n(\mathbb{C})$, $E = E_0 \oplus E_1$: (Peón-Nieto, 2024) identifies components without very stable Higgs bundles.

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- (Recall $E = L_0 \oplus \cdots \oplus L_{n-1}$, divisors $\varphi_j : L_{j-1} \to L_j \otimes K_C$).

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- Conjecture 8.2 in (Hausel–Hitchin, 2022).

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- **Main idea:** $\mathbb{C}^\times$-action on Hecke transformation spaces (Gr_G and $\mathrm{Gr}_G^{\varphi_c}$).

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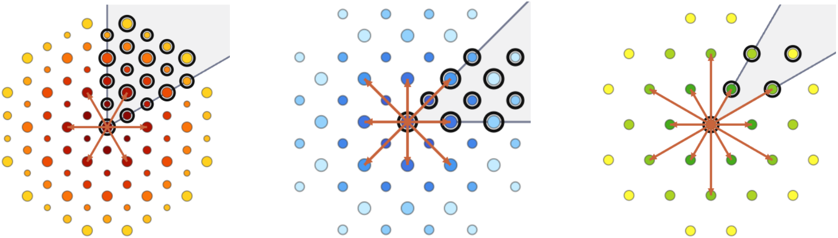

$G = \mathrm{PGL}_3(\mathbb{C})$ $G = \mathrm{SO}_5(\mathbb{C})$ $G = G_2$

Hecke transformation by $\mu \in X_*(T)$ gives another fixed point (E', φ') of Borel type with

$$\mu(E', \varphi') = \mu(E, \varphi) - \mu \cdot c.$$

Strategy III: $\mathbb{C}^\times$-orbits in Gr_G

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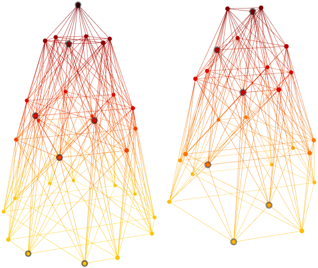

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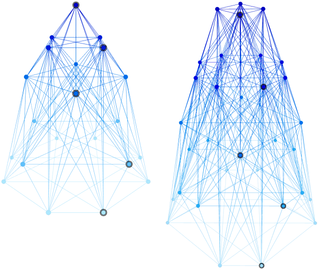

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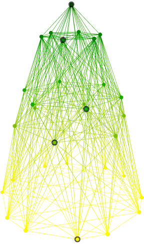

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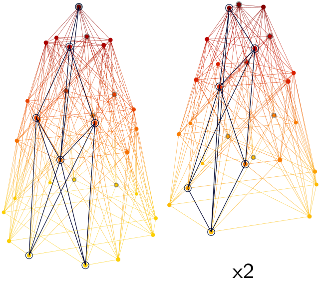

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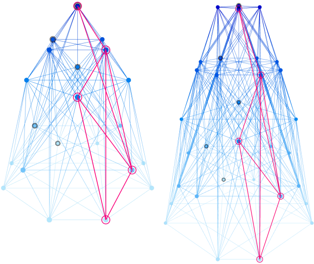

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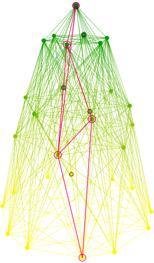

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- Thus (E,φ) is very stable.

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Proposition

Fixed points $(E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$ characterised by:

- E reduces to E_{G_0} a G_0-bundle.
- $\varphi \in H^0((E_{G_0}(\mathfrak{g}_j) \cap \mathrm{SPE}(\mathfrak{g})) \otimes K_C(D))$ for $j \neq 0$.
- $Q_i \in \bigsqcup_{w \in W_{G_0} \backslash W} (E_{G_0}|_{c_i} \cdot w \cdot B)/B \subseteq E|_{c_i}/B$.

Parabolic fixed points of Borel type

- Borel type: $(T, \mathfrak{g}_1 = \bigoplus_{j=1}^r \mathfrak{g}_{\beta_j})$ forces

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Theorem (G. 2026, arXiv:2606.16880)

A smooth fixed point $(E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{s\mathbb{C}^\times}$ of Borel type is very stable if and only if $w(E, \varphi, Q)$ is reduced.

Strategy

- Hecke transformations at $c_i \in D$ given by (affine Springer fibres in) **affine flag variety** $\mathrm{Fl}_G := G((z))/\mathcal{I}$ where $\mathcal{I} \subseteq G[[z]]$ is the **standard Iwahori subgroup**.

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- Similar strategy using Fl_G instead of Gr_G.
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Non-regular case (non-parabolic)

- More general $(G_0, \mathfrak{g}_1)$. **Toledo invariant**

$$-(2g-2)\,\mathrm{rank}_T(G_0, \mathfrak{g}_1) \leqslant \tau(E, \varphi) \leqslant 0$$

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Theorem (García-Prada–G. 2026)

Suppose that

$$\tau(E,\varphi) < 2B^*(\beta,\beta)(g-1)\left(\frac{-\operatorname{rank}_T(\varphi)}{\hat{B}^*(\hat{\beta},\hat{\beta})} + \dim\mathfrak{g}_1 - \dim\hat{\mathfrak{g}}_1\right).$$

Then (E,φ) is not very stable.

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Theorem (García-Prada–G. 2026)

Suppose that $\mathfrak{g} \neq \mathfrak{sl}_2(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C})$, that ${G_0}^e$ as above is reductive and

$$\tau(E, \varphi) = -(2g-2)\,\mathrm{rank}_T(G_0, \mathfrak{g}_1).$$

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Additional questions I

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- (Hausel; G. 2026) If $\mu(E,\varphi) = \mu \cdot c$, pulled from $\mathfrak{t}/W_\mu \to \mathfrak{t}/W$.

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The dual BBB-brane is given by

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- (G. 2026) checks it for $G = \mathrm{PGL}_n(\mathbb{C})$ in parabolic setting, with $\mathrm{supp}(w(E,\varphi,Q)) \subseteq D$.

Happy birthday, Nigel!