Very Stable
$U(p,q)$-Higgs Bundles

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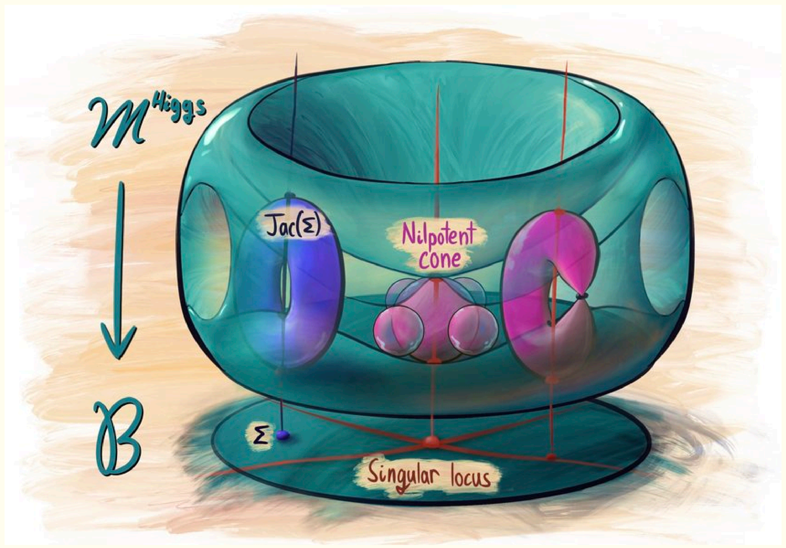
Summary

This project focuses on the study of properties of moduli spaces of Higgs bundles in relation to real forms. More specifically, we investigate the notion of being very stable on the moduli space for the real form $U(p,q) \subset GL(p+q,\mathbb{C})$.

Higgs bundles and the moduli space

Let X be a compact Riemann surface of genus $g \geq 2$. A **Higgs bundle** is a pair (E,Φ) consisting of a rank n holomorphic vector bundle E over X, together with a holomorphic map $\Phi : E \to E \otimes K$, where K is the canonical line bundle over X (the holomorphic cotangent bundle).

These objects were introduced by Hitchin [3] in his study of the self-duality equations over X, and they have played an important role in several areas such as hyperkähler geometry, surface group representations, integrable systems or mirror symmetry and Langlands duality.



(Picture by Elliot Kienzle [4])

There are notions of *stability* and *semistability* for Higgs bundles, arising from Geometric Invariant Theory. The importance is that, fixing $n = \operatorname{rank}(E)$ and $d = \deg(E)$, the set of isomorphism classes of stable Higgs bundles $\mathcal{M}^s(n,d)$ is a complex smooth variety of dimension $2+2n^2(g-1)$, called the **moduli space of stable Higgs bundles**, sitting inside a larger complex quasi-projective variety $\mathcal{M}(n,d)$, the **moduli space of semistable Higgs bundles** [5].

The $\mathbb{C}^*$-action and very stable bundles

We will fix the rank n and the degree d and denote $\mathcal{M} := \mathcal{M}(n,d)$. The space $\mathcal{M}$ carries a $\mathbb{C}^*$-action, with $\lambda \in \mathbb{C}^*$ acting by scalar multiplication on the Higgs field: $(E,\Phi) \mapsto (E,\lambda\Phi)$. The fixed points of this action are of the form:

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_{k-1},$$

with $\Phi(E_j) \subset E_{j+1} \otimes K$. In particular, they are *nilpotent*: $\Phi^k \equiv 0$. There is a natural decomposition of the moduli space $\mathcal{M}$ in vector spaces, called the **Białynicki-Birula decomposition**. It is given by the **upward flows** of fixed points $\mathcal{E} \in \mathcal{M}^{\mathbb{C}}$:

$$W^+_{\mathcal{E}} = \left\{ (E,\Phi) \in \mathcal{M} : \lim_{\lambda \to 0} (E,\lambda\Phi) = \mathcal{E} \right\}.$$

These can be shown to be isomorphic to a vector space, and every point in $\mathcal{M}$ flows to a fixed point, hence belongs in one of these spaces. We say that a fixed point $\mathcal{E} \in \mathcal{M}^{\mathbb{C}}$ is **very stable** if there are no nilpotent elements in $W^+_{\mathcal{E}}$ other than $\mathcal{E}$. This turns out to be equivalent to $W^+_{\mathcal{E}}$ being closed inside $\mathcal{M}$.

We pay special attention to the fixed points where all the summands are line bundles, that is $E = L_0 \oplus \cdots \oplus L_{n-1}$ with $\operatorname{rank}(L_i) = 1$. If we denote $b_i := \Phi|_{L_{i-1}} : L_{i-1} \to L_i \otimes K$, a core result from [2] is:

Theorem 1. *Smooth fixed points $(E = L_0 \oplus \dots L_{n-1}, \Phi)$ are very stable if and only if every zero of the map $b_{n-1} \circ \cdots \circ b_1 : L_0 \to L_{n-1} \otimes K^{n-1}$ is simple.*

The proof makes heavy use of a technique known as Hecke transformations.

Hecke transformations

Given a holomorphic vector bundle E over X, it is possible to obtain a new one by taking a vector subspace of the fiber at a point $x \in X$, and, informally, only considering the sections of E that *pass through* that subspace. This is known as **Hecke transformation**. The resulting bundle only differs from E near x.

More precisely, fixing the subspace $V < E|_x$, the Hecke transformation E' is given by the short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E|_x/V \otimes \mathcal{O}_x \longrightarrow 0.$$

For example, if $E = E_1 \oplus E_2$ and $V = E_1|_x < E|_x$, we are just taking the sections that vanish at $E_2|_x$, hence belonging completely in $E_1|_x$, and indeed the resulting bundle is $E' = E_1 \oplus E_2(-x)$.

For a Higgs bundle (E,Φ), if the space V is invariant by $\Phi|_x$, one also gets a Higgs field Φ' such that (E',Φ') is a new Higgs bundle.

Real forms

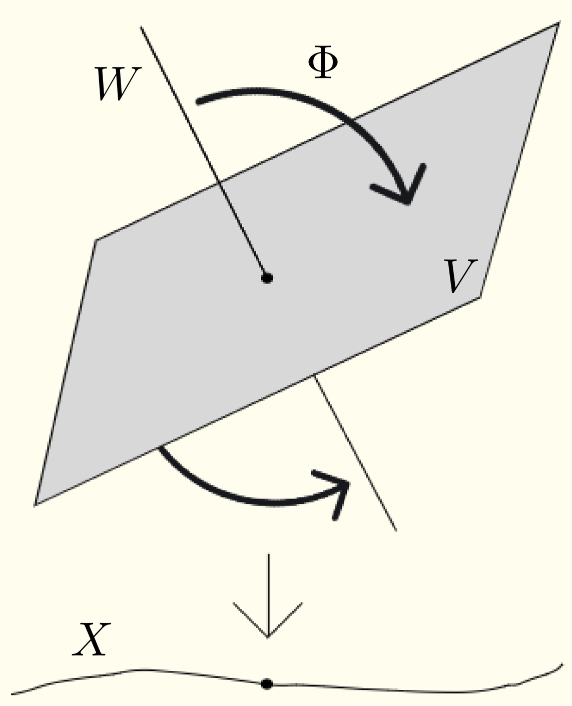
Consider a complex lie algebra $\mathfrak{g}^c$ which is either semisimple or $\mathfrak{gl}(n,\mathbb{C})$. A **real form** is a real Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}^c$ such that $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$. Here are the real forms for the classical Lie algebras:

$\mathfrak{g}^c$	$\mathfrak{g}$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R})$ $\mathfrak{su}(p,q)$, for $p+q=n$ $\mathfrak{su}^*(n)$ if n is even
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(p,q)$
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{sp}(2p,2q)$
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(p,q)$, $\mathfrak{so}^*(2n)$

Real forms have a **Cartan decomposition**, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, from which it is possible to define **G-Higgs bundles**, where G is a real Lie group whose Lie algebra is $\mathfrak{g}$.

$U(p,q)$-Higgs bundles

The subgroup of $GL(p+q,\mathbb{C})$ consisting of the linear invertible operators that preserve an hermitian form of type (p,q), known as $U(p,q)$, is a real form. We can apply the definition of Higgs bundles of real forms to arrive at the following:



Definition. **A $U(p,q)$-Higgs bundle** is a pair (E,Φ), where $E = V \oplus W$ is the sum of holomorphic vector bundles of ranks p and q, and $\Phi \in H^0(\operatorname{End}(E) \otimes K)$ is such that $\Phi(V) \subset W \otimes K$ and $\Phi(W) \subset V \otimes K$.

Example. The fixed points of the $\mathbb{C}^*$-action: let $V := E_0 \oplus E_2 \oplus \dots$ and $W := E_1 \oplus E_3 \oplus \dots$. Since $\Phi(E_j) \subset E_{j+1}$, then $\Phi(V) \subset W \otimes K$ and $\Phi(W) \subset V \otimes K$.

From the definition, $U(p,q)$-Higgs bundles may be regarded as a special kind of $GL(p+q,\mathbb{C})$-Higgs bundles. In fact, as shown in [1], the moduli space $\mathcal{M}_{U(p,q)} \subset \mathcal{M}$ sits inside as the fixed points of the involution $(E,\Phi) \mapsto (E,-\Phi)$. This once again shows that fixed points $\mathcal{E}$ of the $\mathbb{C}^*$-action are $U(p,q)$-Higgs bundles. Moreover, the intersection $W^{2+}_{\mathcal{E}} := W^+_{\mathcal{E}} \cap \mathcal{M}_{U(p,q)}$ is a $\mathbb{C}^*$-invariant vector subspace.

Definition. A fixed point $\mathcal{E} \in \mathcal{M}^{\mathbb{C}^*}$ is **$U(p,q)$-very stable** if the only nilpotent element on the subspace $W^{2+}_{\mathcal{E}}$ is $\mathcal{E}$. Otherwise, we say it is **$U(p,q)$-wobbly**.

Results

Very stable fixed points are $U(p,q)$-very stable. However, the converse need not be true. Our goal is to shed some light on this classification for fixed points $E = L_0 \oplus \cdots \oplus L_{n-1}$ similar to Theorem 1. Keeping that notation, we have the following:

Remark. *If $b_{n-1} \circ \cdots \circ b_1$ has no multiple zeroes, then $\mathcal{E}$ is $U(p,q)$-very stable.*

Remark. *If $n = 2$, $\mathcal{E}$ is $U(p,q)$-very stable.*

Proposition. *If $n \geq 3$ and $b_{n-1} \circ \cdots \circ b_2$ never vanishes, $\mathcal{E}$ is $U(p,q)$-very stable.*

Proposition. *If there are some indices $i \not\equiv j \mod 2$ with $b_i(x) = b_j(x) = 0$ at a given point $x \in X$, then $\mathcal{E}$ is $U(p,q)$-wobbly.*

Proposition. *If $b_{n-2} \circ \cdots \circ b_2$ has a multiple zero $x \in X$, then $\mathcal{E}$ is $U(p,q)$-wobbly.*

The first remark follows from Theorem 1 and the second from observing that nilpotent $U(p,q)$-Higgs bundles of rank 2 are fixed points. The first proposition is proven by constructing certain section of the Hitchin map which turns out to be the whole upward flow, from which we see that there are no extra nilpotents. The last two propositions are proven by applying Hecke transformations to create a *Hecke curve* of nilpotent elements in the upward flow starting from the fixed point.

References

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